Faculty of Graduate Study

Dynamics of Rational Difference Equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}$$

Using Mathematical And Computational Approach

Prepared by

MUNA ABU ALHALAWA

Supervised by

Prof. MOHAMMAD SALEH

M.Sc. Thesis
Birzeit University
Palestine
2009

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This thesis was submitted in partial fulfillment of the requirements for the master's degree in Mathematics From the Faculty of Graduate Studies at Birzeit University, Palestine

BIRZEIT UNIVERSITY
MATHEMATICS DEPARTMENT

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled

Dynamics of Rational Difference Equation

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Using Mathematical And Computational Approach

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MUNA ABU ALHALAWA

in partial fulfillment of the requirements for the degree of Master.
This thesis was defended successfully on May, 2009

Committee Members		Signature
1. Prof. Mohammad Saleh	**Head of Committee**	....
2. Dr. Marwan Aloqeili	**Internal Examiner**	....
3. Dr. Tahseen Mughrabi	**External Examiner**	....

Dedication

Thankfully I dedicate this thesis to all those who contributed to its success. I dedicate it to my beloved family: father, mother, brothers and sisters who through their encouragement, patience and support enabled me to continue my study and get this degree.

I am also very grateful to my school teachers and university teachers and professors who were like a candle to me and that enlightened my mind and life.

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I would like also to express my thanks to Dr. Tahseen Mughrabi. He was my mentor ever since I was a freshman. More than just a mentor, he became a very good friend.

Finally I dedicate this thesis to all those who supported me, believed in me. There are too many important friends, they know who they are, for without their love and support I would not have been able to reach this point.

THANK YOU ALL

Abstract

The main goal of this thesis is to investigate the periodic character, invariant intervals, oscillation and global stability and other new results of all positive solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \; n = 0, 1, 2, ...$$

where the parameters α, β, A, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$ are non-negative real numbers with the solution is defined and $k \in \{1, 2, 3, ...\}$.

We give a detailed description of the semi-cycles of solutions, and determine conditions under which the equilibrium points are globally asymptotically stable.

In particular, our monograph is a generalization of the rational difference equation that was investigated in [15].

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DECLARATION

I certify that this thesis, submitted for the degree of Master of Science to the Department of Mathematics in Birzeit University, is of my own research expect where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

Muna Abu Alhalawa   Signature................
May 14, 2009

Contents

Introduction

The dynamical system is the study of phenomena that evolve in space and/ or time by looking at the dynamic behavior or the geometrical and topological properties of the solution, whether a particular system comes from Economics, Biology, Physics, Chemistry, or even Social Science. The dynamical system is the subject that provides the mathematical tools for it's analysis.
Dynamical system in point of view of mathematics is a system whose behavior at given time depends, in some sense, on it's behavior at one or more previous times.

An equation which express a value of a sequence as a function of the other terms in the sequence is called a difference equation.
In particular, an equation which expresses the value x_n of a sequence a_n as a function of the term a_{n-1} is called a first order difference equation. If we can find a function f such that $a_n = f(n)$, $n = 1, 2, 3, ...$ then we will have solved the difference equation.

This thesis consists mainly of 5 chapters, where Chapter 1 deals with first order difference equations. We focus on the equilibrium points and their stability, the Cobweb Diagram and periodic points. Chapter 2 deals with difference equations of higher order. We focus on the solution of k^{th} order homogeneous linear difference equations with constant coefficients and the solution of nonlinear difference equation, equilibrium points of difference equations, the linearization and the global stability theorems of non linear difference equations. In Chapter 3 we will study the Dynamics of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \; n = 0, 1, 2, ... \tag{0.1}$$

In Chapter 4 we will mention the special cases of Eq.(0.1) and study some of them.
Finally chapter 5 presents the Matlab codes of all figures in this thesis.

Eq.(0.1) was studied by *G.LADAS* in [15], when $k = 1$. He studied the equilibrium points and the local and global stability of the solution of the equation.

The aim of this thesis is to study: equilibrium points, local stability and global stability, periodic solution, semicycles and boundedness of the solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \ n = 0, 1, 2, ...$$

where the parameters α, β, A, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$ are non-negative real numbers with the solution is defined and $k \in \{1, 2, 3, ...\}$.

We are particulary interested in the asymptotic behavior of the solutions, that is the behavior of the solution as $n \to \infty$.

Chapter 1

Dynamics of First Order Difference Equations

1 Dynamics of First Order Difference Equations

1.1 Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time. In difference equations the term x_{n+1} is related to the term x_n and the relation is expressed in the difference equation

$$x_{n+1} = f(x_n) \tag{1.1}$$

starting from a point x_0, we can generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...$$

and for convenience we use the notation

$$\begin{aligned} f^2(x_0) &= f(f(x_0)) = x_2 \\ f^3(x_0) &= f(f(f(x_0))) = x_3 \\ &. \\ &. \\ &. \\ f^n(x_0) &= x_n \end{aligned}$$

where $f(x_0)$ is called the first iterate of x_0 under f, and $f^2(x_0)$ is called the second iterate of x_0 under f, and more generally $f^n(x_0)$ is the n-th iterate of x_0 under f.
Thus we can have

$$x_{n+1} = f^{n+1}(x_0) = f(f^n(x_0)) = f(x_n)$$

This iterative procedure is an example of a discrete dynamical system. In particular, we can find out the solution of linear first order difference equation by forward iteration with initial condition x_0.
For example, let us consider the simplest case of the linear difference equation

$$x_{n+1} = ax_n$$

with the initial condition x_0, so we get the solution by forward iteration with the initial condition x_0 as follows:

$$\begin{aligned}
x_1 &= ax_0 \\
x_2 &= ax_1 = a(ax_0) = a^2x_0 \\
x_3 &= ax_2 = a^3x_0 \\
. \\
. \\
. \\
x_n &= a^nx_0
\end{aligned}$$

We can notice that the limiting behavior of the solution of equation

$$x_{n+1} = ax_n$$

is as follow:

1. If $|a| < 1$, then $\lim\limits_{n\to\infty} x_n = 0$
2. If $a > 1$, then $\lim\limits_{n\to\infty} x_n = \infty$.
3. if $a < -1$, then $\lim\limits_{n\to\infty} x_n$ does not exist.
4. If $a = 1$, then every point is an equilibrium point.
5. If $a = -1$, then $x_n = \begin{cases} x_0 & if \quad n \ is \ even \\ -x_0 & if \quad n \ is \ odd \end{cases}$

   or $x_n = (-1)^n x_0$

1.2 The Equilibrium Points

Let us consider the difference equation

$$x_{n+1} = f(x_n) \tag{1.2}$$

Definition 1.1. A point $\overline{x}$ is said to be an **equilibrium** point if it is a **fixed** point of the map f of the Eq.(1.2) ; i.e; if $f(\overline{x}) = \overline{x}$.

Example 1.1. *Determine the fixed points of the following function*

$$f(x) = x^2 - 4x + 6$$

Solution: We can find the fixed points by solving the following equation:

$$f(x) = x$$

then, we get

$$x^2 - 4x + 6 = x$$

hence

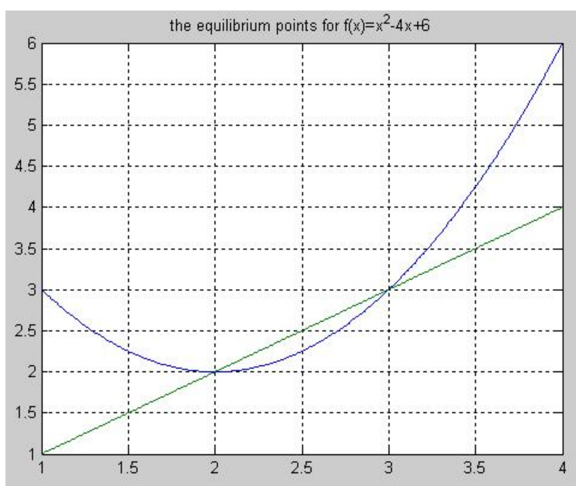
$$x^2 - 5x + 6 = 0$$

then

$$(x - 2)(x - 3) = 0$$

hence, there are two fixed points

$$\overline{x} = 2 \text{ and } \overline{x} = 3$$



1.3 Stability Theorem

One of the main objectives in the theory of dynamical systems is the study of the behavior of its solution near the equilibrium point, such investigation is called **Stability theory**. To do this investigation, we begin by introducing the basic notions of stability.

Definition 1.2. Let $f : I \longrightarrow I$ where I is an interval in the set of real numbers $\Re$ and $\overline{x}$ be an equilibrium point of the difference equation

$$x_{n+1} = f(x_n) \tag{1.3}$$

then

1. The equilibrium point $\overline{x}$ of Eq.(1.3) is called **locally stable** if for every ϵ, there exists δ such that if

   $$|x_0 - \overline{x}| < \delta$$

   then

   $$|x_n - \overline{x}| < \epsilon$$

   for all n ≥ 1, and all $x \in I$.

2. The equilibrium point is called **unstable** if it is not stable.

3. The equilibrium point $\overline{x}$ of Eq.(1.3) is called **lacally asymptotically stable** or (asymptotically stable) if it is stable and if there exists $\gamma > 0$ such that if

   $$|x_0 - \overline{x}| < \gamma$$

   then

   $$\lim_{n\to\infty} x_n = \overline{x}$$

4. The equilibrium point $\overline{x}$ of Eq.(1.3) is called **global attractor** if for every

   $$x_0 \in I$$

   then

   $$\lim_{n\to\infty} x_n = \overline{x}$$

5. The equilibrium point $\overline{x}$ of Eq.(1.3) is called **globally asymptotically stable** (or globally stable) if it is stable and is global attractor.

6. The equilibrium point $\overline{x}$ of Eq.(1.3) is called **repeller** if there exists $r > 0$ such that if $x_0 \in I$ and

   $$|x_0 - \overline{x}| < r$$

   then there exists $N \geq 1$ such that

   $$|x_N - \overline{x}| > r$$

Clearly, a repeller is an unstable equilibrium point.

1.4 The Cobweb Diagram

One of the important graphical method for analyzing the stability of equilibrium points for (x_n, x_{n+1}) is the cobweb diagram, since $x_{n+1} = f(x_n)$. We draw a graph of f, we can choose our initial point x_0, then we can find x_1 from the graph. This could be done by drawing a vertical line from the point x_0, so that it also intersects the graph of f at (x_0, x_1). Next draw a horizontal line from (x_0, x_1) to meet the diagonal line $y = x$ at the point (x_1, x_1). Now again a vertical line drawn from (x_1, x_1) will meet the graph of f at (x_1, x_2), continuing this process we can evaluate all the points in the orbit of x_0, namely, the set $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$ or equivalently $\{x_0, f(x_0), f^2(x_0), \cdots, f^n(x_0), \cdots\}$.

Definition 1.3. Let $\mu > 0$, then the difference equation

$$x_{n+1} = \mu x_n(1 - x_n)$$

is called discrete Logistic difference equation, and the function

$$f_\mu(x) = \mu x(1 - x)$$

is called Logistic Map.

Example 1.2. *: Consider the difference equation*

$$x_{n+1} = \mu x_n(1 - x_n)$$

for $\mu = 2$ and $\mu = 3.6$

1. *Find the equilibrium (fixed) points.*

2. *Determine the stability of the equilibrium points by using Cobweb diagram.*

Solution: To find the fixed points of f_μ, we solve the equation

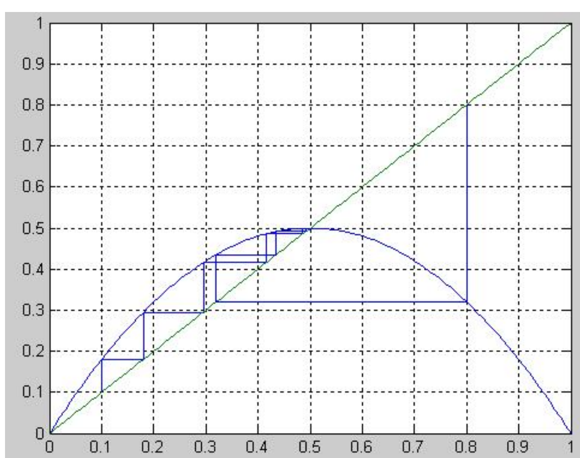
$$\mu x(1 - x) = x$$

. *This yields two equilibrium points: $\overline{x}_1 = 0$, and $\overline{x}_2 = \frac{\mu-1}{\mu}$.*

- *When $\mu = 2$, then the two equilibrium points are $\overline{x}_1 = 0$ and $\overline{x}_2 = \frac{1}{2}$. The stability can be achieved from cobweb diagram.*

  Figure.A. shows the cobweb diagram, from which we can see that the equilibrium point $\overline{x}_2$ is asymptotically stable.

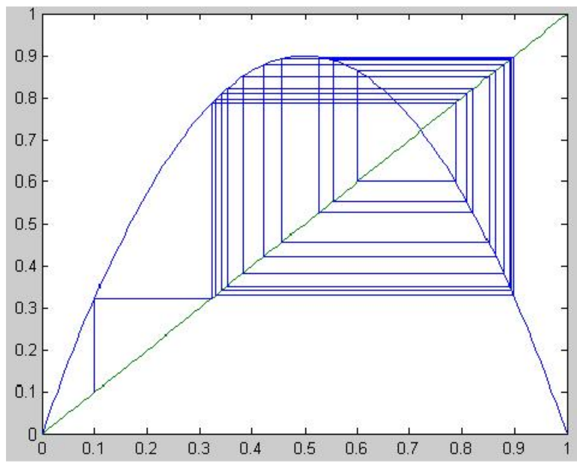
Figure.A.



- *When* $\mu = 3.6$*, then the two equilibrium points are* $\overline{x}_1 = 0$ *and* $\overline{x}_2 = 0.7222$*. The stability can be achieved from cobweb diagram.*

  Figure.B. shows the cobweb diagram, from which we can see that the equilibrium point $\overline{x}_2$ *is unstable.*

Figure.B.



1.5 Criteria for Stability

In this section, we are going to introduce some powerful criteria for local stability of equilibrium(fixed) points. Equilibrium points are divided into two types: **hyperbolic** and **non hyperbolic**. A fixed point $\overline{x}$ of a map f is said to be hyperbolic if $|f'(\overline{x})| \neq 1$. Otherwise it is non hyperbolic.

Theorem 1.4. *[11] (Criteria for Stability) Let $\overline{x}$ be a hyperbolic fixed point of a map f, where f is continuously differentiable at $\overline{x}$. The following statements then holds true:*

1. *If $|f'(\overline{x})| < 1$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is asymptotically stable.*
2. *If $|f'(\overline{x})| > 1$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is unstable.*

Theorem 1.5. *[11] Suppose that for an equilibrium point $\overline{x}$ of Eq.(1.2), $f'(\overline{x}) = 1$. The following statements then holds true:*

1. *$f''(\overline{x}) \neq 0$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is unstable.*
2. *$f''(\overline{x}) = 0$ and $f'''(\overline{x}) > 0$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is unstable.*
3. *If $f''(\overline{x}) = 0$ and $f'''(\overline{x}) < 0$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is asymptotically stable.*

1.6 Periodic Points

In studying the dynamical system its important to study its periodicity. An example: the motion of the pendulum is periodic.

Definition 1.6. Let b be in the domain of f. Then:

1. b is called a periodic point of f in Eq.(1.2) if for some positive integer k, $f^k(b) = b$. Hence a point is k-periodic if it is a fixed point of f^k. The periodic orbit of b is

$$O(b) = \{b, f(b), f^2(b), ..., f^{k-1}(b)\}$$

and it's often called a k-orbit.

2. b is called eventually k-periodic if for some positive integer m, $f^m(b)$ is a k-periodic point; in other words $f^{m+k}(b) = f^m(b)$.

Example 1.3. *Consider the difference equation generated by the tent function*

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{1.4}$$

We can first obtain that the periodic points of period 2 *are the fixed points of* T^2 *. It is easy to verify that* T^2 *is given by*

$$T^2(x) = \begin{cases} 4x & \text{if } 0 \le x < \frac{1}{4}, \\ 2(1-2x) & \text{if } \frac{1}{4} \le x < \frac{1}{2}, \\ 4(x-\frac{1}{2}) & \text{if } \frac{1}{2} \le x < \frac{3}{4}, \\ 4(1-x) & \text{if } \frac{3}{4} \le x \le 1. \end{cases} \tag{1.5}$$

There are four equilibrium points : $0, 0.4, \frac{2}{3},$ *and* 0.8, *two of which* 0 *and* $\frac{2}{3}$, *are equilibrium points for* T. *Hence* $\{0.4, 0.8\}$ *is the only* 2- *cycle of* T.

Definition 1.7. Let b be a k-periodic point of f. Then b is:

1. Stable if it is a stable fixed point of f^k.

2. Asymptotically stable if it is an asymptotically stable fixed point of f^k.

3. Unstable if it is an unstable fixed point of f^k.

Chapter 2
Difference Equations of Higher Order

2 Linear Difference Equations of Higher Order

2.1 General Theory of Linear Difference Equations

The normal form of k^{th} order nonhomogeneous linear difference equation is given by:

$$x_{n+k} + p_1(n)x_{n+k-1} + p_2(n)x_{n+k-2} + \cdots + p_k(n)x_n = g(n) \tag{2.1}$$

where $p_i(n)$ and $g(n)$ are real valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$. If $g(n) = 0$, then the Eq.(2.1) is said to be a homogeneous equation. Now the equation:

$$x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + \cdots + p_k x_n = 0 \tag{2.2}$$

is called linear difference equation of k^{th} order with constant coefficients.

The sequence x_n is said to be a solution of Eq.(2.1) if it satisfies the equation. If we specify our initial conditions of the equation, this lead us to the initial value problem

$$x_{n+k} + p_1(n)x_{n+k-1} + p_2(n)x_{n+k-2} + \cdots + p_k(n)x_n = g(n) \tag{2.3}$$

$$x_{n_0} = a_0,\ x_{n_0+1} = a_1$$

$$\vdots$$

$$x_{n_0+k-1} = a_{k-1} \tag{2.4}$$

where a_i's are real numbers.

Example 2.1. *Consider the 2nd order homogeneous difference equation*

$$x_{n+2} = 4x_{n+1} + 5x_n \tag{2.5}$$

where $x_0 = 1$, $x_1 = 2$, then we can find x_2, x_3:

$$\begin{aligned} x_2 &= 4x_1 + 5x_0 = 13 \\ x_3 &= 4x_2 + 5x_1 = 62 \end{aligned}$$

and by the same method, we can get x_4, x_5, $\cdots$.
So if we have the initial conditions, then we can find the whole solution of our difference equation.

Theorem 2.1. *[11] The initial value problems of Eq.(2.3) and Eq.(2.4) have a unique solution x_n.*

Definition 2.2. The functions $f_1(n)$,$f_2(n)$,$\cdots$,$f_r(n)$ are said to be linearly dependent for $n_0 \leq n$ if there are nonzero constants a_1,a_2,$\cdots$,a_r such that

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

.

The negation of linear dependence is linear independence. Then the set of functions $f_1(n)$,$f_2(n)$,$\cdots$,$f_r(n)$ are said to be linearly independent if whenever

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

for all $n_0 \leq n$, then we must have $a_1 = a_2 = \cdots = a_r = 0$.

Definition 2.3. A set of k linearly independent solution of Eq.(2.2) is called a fundamental set of solutions.

Theorem 2.4. *(The Fundamental Theorem) [11] If $p_k \neq 0$ is non zero for all k, then Eq.(2.2) has a fundamental set of solutions.*

Theorem 2.5. *(Superposition Principle) [11] If $x_1(n)$,$x_2(n)$,$\cdots$,$x_r(n)$ are solutions of Eq.(2.2), then also*

$$x(n) = a_1 x_1(n) + a_2 x_2(n) + \cdots + a_r x_r(n)$$

is a solution of Eq.(2.2), where a_1,a_2,$\cdots$,a_r are real numbers.

Example 2.2. *Consider the third order homogeneous difference equation*

$$x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n = 0$$

Where the functions 1, n, n^2 form the fundamental set of solutions of the equation.

We can verify that the fundamental set forms a solution of the equation by substituting $x_n = 1$, $x_n = n$, $x_n = n^2$ into the equation.

From superposition principle we can say that

$$x_n = c_1 + c_2 n + c_3 n^2$$

where c_1,c_2,c_3 are real numbers, is also a solution of the equation, which can be done easily.

In the remaining of this section we will give all possible solutions of Eq.(2.2). The solutions of Eq.(2.1) have been investigated in [11].

2.2 Solution of k^{th} order homogeneous linear difference equation with constant coefficients

Now, consider the k^{th} order homogeneous linear difference equation (2.2) where the p_i's are constants and $p_k \neq 0$. Define λ to be a characteristic root of Eq.(2.2), then λ^n is a solution of Eq.(2.2). Substitute λ^n into Eq.(2.2), we obtain:

$$\lambda^k + p_1 \lambda^{k-1} + \cdots + p_k = 0 \tag{2.6}$$

which is called the **characteristic equation** of Eq.(2.2)

The general solution of Eq.(2.2) has different forms depending on λ's.

1. Distinct roots
   Suppose that the characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct.i.e.

   $$|\lambda_1| \neq |\lambda_2| \neq \cdots |\lambda_k|$$

   So the general solution is:

   $$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_k \lambda_k^n$$

2. Repeated Roots

   $$\lambda_1 = \lambda_2 = \cdots = \lambda_m, \ 2 \leq m \leq k$$

Then the general solution of difference equation(2.2) is given by:

$$x_n = c_1\lambda_1^n + c_2 n\lambda_2^n + \cdots + c_m n^{m-1}\lambda_m^n + c_{m+1}\lambda_{m+1}^n + \cdots + c_k\lambda_k^n$$

3. The absolute value of the roots are equal
   i.e.
   $$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k|$$
   - The characteristic roots are equal
     the general solution is:
     $$x_n = c_1\lambda_1^n + c_2 n\lambda_2^n + \cdots + c_k n^{k-1}\lambda_k^n$$
   - The characteristic roots are not equal
     $$\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda$$
     and
     $$\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_k = -\lambda$$
     The general solution is given by:
     $x_n = (c_1 + c_2 n + c_3 n^2 + \cdots + c_m n^{m-1})\lambda^n +$
     $(c_{m+1} + c_{m+2} n + c_{m+3} n^2 + \cdots + c_k n^{k-m-1})(-1)^n\lambda^n$
4. Some of roots are complex
   Assume that
   $$\lambda_1 = \alpha + i\beta$$
   and
   $$\lambda_2 = \alpha - i\beta$$
   and that $\lambda_3, \lambda_4, \cdots, \lambda_k$ are all real and distinct such that
   $$|\lambda_3| > |\lambda_4| > \cdots > |\lambda_k|$$
   where
   $$\begin{aligned}\lambda_1 &= \alpha + i\beta\\ &= re^{i\phi}\\ &= r(cos\ \phi + i\ sin\ \phi)\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= \alpha - i\beta \\ &= re^{-i\phi} \\ &= r(cos\ \phi - i\ sin\ \phi)\end{aligned}$$

Then the general solution of Eq.(2.2) is given by:

$$\begin{aligned}x_n &= c_1 r^n e^{in\phi} + c_1 r^n e^{-in\phi} + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= c_1 r^n(\cos\ n\phi + i\ sin\ n\phi) + c_2 r^n(\cos\ n\phi - i\ sin\ n\phi) + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= (c_1 + c_2)r^n \cos\ n\phi + (c_1 - c_2)r^n i\ \sin\ n\phi + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= r^n[(c_1 + c_2)\cos\ n\phi + (c_1 - c_2)i\ \sin\ n\phi] + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= r^n[a_1 \cos\ n\phi + a_2\ \sin\ n\phi] + c_3\lambda_3^n + \cdots + c_k\lambda_k^n\end{aligned}$$

where $a_1 = c_1 + c_2$ and $a_2 = (c_1 - c_2)i$. Now, Let

$$\cos\ \omega = \frac{a_1}{\sqrt{a_1^2 + a_2^2}},\ \sin\ \omega = \frac{a_2}{\sqrt{a_1^2 + a_2^2}},\ \omega = \arctan(\frac{a_2}{a_1})$$

The solution will be

$$\begin{aligned}x_n &= r^n\sqrt{a_1^2 + a_2^2}[\cos\ \omega\ \cos\ n\phi + \sin\ \omega\ \sin\ n\phi] + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= r^n\sqrt{a_1^2 + a_2^2}\cos\ (n\phi - \omega) + c_3\lambda_3^n + \cdots + c_k\lambda_k^n \\ &= Ar^n \cos\ (n\phi - \omega) + c_3\lambda_3^n + \cdots + c_k\lambda_k^n\end{aligned}$$

where

$$A = \sqrt{a_1^2 + a_2^2},\ r = \sqrt{\alpha^2 + \beta^2},\ \phi = \arctan(\frac{\beta}{\alpha})$$

2.3 Solution of k^{th} order nonhomogeneous linear difference equations with constant coefficients

The main idea of solving such difference equations is to find particular solution in addition to homogeneous solution, and there are some techniques discussed in this manner in [2].

Example 2.3. *Find the general solution of*

$$x_{n+2} - 5x_{n+1} + 6x_n = 4^n n^2$$

Solution:
Let x_0, x_1 be two initial conditions. Then

$$x_n = x_{hn} + x_{pn}$$

where x_n is the general solution, x_{hn} is the homogeneous solution, and x_{pn} is the particular solution.
To find the homogeneous solution: solve the characteristic equation:

$$r^2 - 5r + 6 = 0$$

$$r^2 - 5r + 6 = (r-2)(r-3) = 0$$

$\Rightarrow$

$$r_1 = 2,\ r_2 = 3$$

Then, the homogeneous solution is:

$$\begin{aligned} x_{hn} &= ar_1^n + br_2^n \\ &= a2^n + b3^n \end{aligned}$$

To find the particular solution, let

$$x_p = c4^n + dn4^n + en^2 4^n$$

substituting this potential solution into the equation and equating coefficients as following

$$x_{pn} = c4^n + dn4^n + en^2 4^n$$

$$x_{pn+1} = c4^{n+1} + d(n+1)4^{n+1} + e(n+1)^2 4^{n+1}$$

$$x_{pn+2} = c4^{n+2} + d(n+2)4^{n+2} + e(n+2)^2 4^{n+2}$$

Hence, we get

$$c4^{n+2}+d(n+2)4^{n+2}+e(n+2)^2 4^{n+2}-5(c4^{n+1}+d(n+1)4^{n+1}+e(n+1)^2 4^{n+1})+$$

$$+6(c4^n + dn4^n + en^2 4^n)$$

$$= 4^n n^2$$

after some simple algebraic calculations, we get

$\Rightarrow 6c = 1 \Rightarrow c = \frac{1}{6}$

$\Rightarrow -2d = 0 \Rightarrow d = \frac{-10}{9}$

$\Rightarrow d - e = 0 \Rightarrow e = \frac{244}{108}$

Thus, the general solution of the equation is:

$$x_n = a2^n + b3^n + \frac{244}{108}4^n - \frac{10}{9}n4^n + \frac{1}{6}n^2 4^n$$

To find the values of the constants a and b the initial conditions x_0, x_1 must be provided.

2.4 Limiting Behavior of Solutions

To simplify our exposition we restrict our discussion to the second order difference equation

$$x_{n+2} + p_1 x_{n+1} + p_2 x_n = 0 \tag{2.7}$$

Suppose that λ_1 and λ_2 are the characteristic roots of the equation. Then we have the following three cases:

Case 1: λ_1 and λ_2 are distinct roots. Then $x_1(n) = \lambda_1^n$ and $x_2(n) = \lambda_2^n$ are two linearly independent solutions of Eq.(2.7) and the general solution is given by :

$$x_n = a_1 \lambda_1^n + a_2 \lambda_2^n$$

Example 2.4. *Consider the equation*

$$x_{n+2} = 3x_{n+1} - 2x_n$$

then the characteristic equation is:

$$\lambda^2 - 3\lambda + 2 = 0$$

The solutions to the quadratic equation are

$$\lambda = 1, \ \lambda = 2$$

and the general solution is

$$x_n = a_1 + a_2 2^n$$

Now assume that

$$|\lambda_1| > |\lambda_2|$$

then λ_1 is called the dominant root and $x_1(n)$ the dominant solution.

The general solution could be written as

$$x_n = \lambda_1^n(a_1 + a_2(\frac{\lambda_2}{\lambda_1})^n)$$

It is clear that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_1\lambda_1^n$ since $\frac{\lambda_2}{\lambda_1} < 1$, then $\lim_{n\to\infty} \frac{\lambda_2}{\lambda_1} = 0$.

We can notice that:

1. If $|\lambda_1| > 1$, then the solution x_n will diverge.

2. If $\lambda_1 = 1$, then the solution x_n will be a constant solution.

3. If $\lambda_1 = -1$, then the solution x_n will be oscillating between two values a_1 and $-a_1$.

4. If $|\lambda_1| < 1$, then the solution x_n will converges to zero.

Case 2: $\lambda_1 = \lambda_2 = \lambda$, then the general solution is given by

$$x_n = \lambda^n(a_1 + a_2 n)$$

Example 2.5. *Consider the equation*

$$x_{n+2} + 4x_{n+1} + 4 = 0$$

then the characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

the solution to the quadratic equation is

$$\lambda = -2$$

and the general solution is

$$x_n = a_1(-2)^n + a_2 n(-2)^n$$

It is obvious that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \lambda^n(a_1 + a_2 n)$*, then*

1. *If* $|\lambda| < 1$, *then the solution will converge to zero since* $\lim_{n\to\infty} n\lambda^n = 0$.

2. *If* $|\lambda| \geq 1$, *then the solution will diverge.*

Case 3: λ_1 and λ_2 are Complex roots; i.e; $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where $\beta \neq 0$. Then the general solution will be

$$x_n = a_1(\alpha + i\beta)^n + a_2(\alpha - i\beta)^n$$

In polar coordinates the complex number $\alpha + i\beta$ could be written as $x_n = re^{i\phi}$ where

$$r = \sqrt{\alpha^2 + \beta^2},\ \alpha = r\cos\phi,\ \beta = r\sin\phi,\ \phi = \arctan(\frac{\beta}{\alpha})$$

then

$$x_n = a_1(r\cos\phi + ir\sin\phi)^n + a_2(r\cos\phi - i\sin\phi)^n$$

so after arrangement we get

$$x_n = r^n(c_1\cos(n\phi) + c_2\sin(n\phi))$$

where $c_1 = a_1 + a_2$ and $c_2 = a_1 - a_2$.

Let

$$\cos\ \omega = \frac{c_1}{\sqrt{c_1^2 + c_2^2}},\ \sin\ \omega = \frac{c_2}{\sqrt{c_1^2 + c_2^2}},\ \omega = \arctan(\frac{c_2}{c_1})$$

then we can write the solution as

$$x_n = Cr^n\cos(n\phi - \omega)$$

where $C = \sqrt{c_1^2 + c_2^2}$.

The solution x_n is oscillating since the cosine function oscillates. But this oscillation have three different cases:

1. If $r < 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie inside the unitary disk and the solution will converge to zero.

2. If $r = 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie on the unitary disk and the solution oscillate in constant magnitude.

3. If $r > 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie outside the unitary disk and the solution will diverge.

Example 2.6. *Consider the equation*

$$x_{n+2} - 2x_{n+1} + 2x_n = 0$$

the characteristic equation is

$$\lambda^2 - 2\lambda + 2 = 0$$

then

$$\lambda_1 = 1 + i \text{ and } \lambda_2 = 1 - i$$

where $r = \sqrt{2}$ *and* $\phi = \arctan(1)$.

The real formed solution is

$$x_n = 2^{\frac{n}{2}}(c_1 \cos(n\phi) + c_2 \sin(n\phi))$$

Then since $r = \sqrt{2} > 1$, *then* λ_1 *and* $\lambda_2 = \overline{\lambda_1}$ *lie outside the unitary disk and the solution will diverge.*

Theorem 2.6. *[11] The following statements hold:*

1. *All solutions of Eq.(2.7) oscillate about zero if and only if the characteristic equation has no positive real roots.*

2. *All solutions of Eq.(2.7) converge to zero if and only if*

$$\max\{|\lambda_1|, |\lambda_2|\} < 1$$

Consider the *second order non homogeneous difference equation*

$$x_{n+2} + p_1 x_{n+1} + p_2 x_n = M \tag{2.8}$$

where M is nonzero. Suppose that $\overline{x}$ is an equilibrium point of such equation, then

$$\overline{x} + p_1\overline{x} + p_2\overline{x} = M$$

solving for $\overline{x}$ we get

$$\overline{x} = \frac{M}{1 + p_1 + p2}$$

But the general equation of the nonhomogeneous equation is

$$x_n = x_{hn} + x_{pn}$$

Where x_{hn} is the solution of the homogeneous equation, and x_{pn} is the particular solution. For this equation we take $x_{pn} = \overline{x}$.

Theorem 2.7. *[11] The following statements holds:*

1. *All solutions of Eq.(2.8) oscillate about $\overline{x}$ if and only if the characteristic homogeneous equation of Eq.(2.7) has no positive real roots.*

2. *All solutions of Eq.(2.8) converges to $\overline{x}$ if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$ *where λ_1 and λ_2 are the real roots of the homogeneous characteristic equation of Eq.(2.7).*

Higher Order Scalar Difference Equations

2.5 Definitions

Here, we list some definitions which will be useful in our investigation.

Definition 2.8. Let I be some interval of real numbers and let

$$f : I_{k+1} \longrightarrow I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \cdots x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}),\ n = 0, 1, \cdots \tag{2.9}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.9. A point $\overline{x}$ is called an equilibrium point of Eq.(2.9) if

$$\overline{x} = f(\overline{x}, \overline{x}, \cdots, \overline{x})$$

that is

$$x_n = \overline{x}, \;\; for \; n \geq 0$$

is a solution of Eq.(2.9), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2.10. Let $\overline{x}$ be an equilibrium point of Eq.(2.9)

1. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **stable** if for every ϵ, there exists δ such that if

   $$x_{-k}, \cdots x_{-1}, x_0 \in I$$

   and

   $$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta$$

   then

   $$|x_n - \overline{x}| < \epsilon$$

   for all $n \geq -k$

2. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **lacally asymptotically stable** if is it stable and if there exists $\gamma > 0$ such that if

   $$x_{-k}, \cdots, x_{-1}, x_0 \in I$$

   and

   $$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$$

   then

   $$\lim_{n \to \infty} x_n = \overline{x}$$

3. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **global attractor** if for every

   $$x_{-k}, \cdots, x_{-1}, x_0 \in I$$

   we have

   $$\lim_{n \to \infty} x_n = \overline{x}$$

4. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **global asymptotically stable** if it is stable and global attractor.

5. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **unstable** if it is not stable

6. The equilibrium point $\overline{x}$ of Eq.(2.9) is called **repeller** if there exists $r > 0$ such that if $x_{-k}, \cdots, x_{-1}, x_0 \in I$ and

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$$

then there exists $N > -k$ such that

$$|x_N - \overline{x}| > r$$

Clearly, a repeller is an unstable equilibrium.

Definition 2.11. Let $a = \frac{\partial f}{\partial x}(\overline{x}, \overline{x})$ and $b = \frac{\partial f}{\partial y}(\overline{x}, \overline{x})$ where $f(x, y)$ is the function in Eq.(2.9) and $\overline{x}$ is the equilibrium of Eq.(2.9). Then the equation

$$z_{n+1} = az_n + bz_{n-k}, \ n = 0, 1, 2, \cdots \tag{2.10}$$

is called linearized equation associated with Eq.(2.9) about the equilibrium point $\overline{x}$, and its characteristic equation is

$$\lambda^{k+1} - a\lambda^k - b = 0. \tag{2.11}$$

Theorem 2.12. *[17]:*(Linearized Stability)

1. *If all the roots of Eq.(2.11) lie in open disk $|\lambda| < 1$, then the equilibrium point $\overline{x}$ of Eq.(2.9) is asymptotically stable.*

2. *If at least one root of Eq.(2.11) has absolute value greater than 1, then the equilibrium $\overline{x}$ of Eq.(2.9) is unstable.*

Theorem 2.13. *[13] Assume that $p, q \in \Re$ and $k \in \{1, 2, 3, \cdots\}$. Then a necessary and sufficient condition for asymptotic stability of the equation*

$$x_{n+1} - px_n - qx_{n-1} = 0, \ n = 0, 1, 2, \cdots$$

is that

$$|p| < 1 - q < 2$$

Theorem 2.14. *[5] Assume* $a, b \in \Re$ *and* $k \in \{1, 2, \cdots\}$. *Then*

$$|a| + |b| < 1 \tag{2.12}$$

is sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} - ax_n + bx_{n-k} = 0, \; n = 0, 1, 2, \cdots \tag{2.13}$$

Suppose in addition that one of the following two cases holds:

1. k *is odd and* $b < 0$.
2. k *is even and* $ab < 0$.

Then Eq.(2.12) is a necessary condition for asymptotic stability of Eq.(2.13).

Theorem 2.15. *[16] The difference equation*

$$y_{n+1} - by_n + by_{n-k} = 0, \; n = 0, 1, 2, ...$$

is asymptotically stable if and only if $0 < |b| < \frac{1}{2}\cos(\frac{k\pi}{k+2})$

Theorem 2.16. *[14] Let* $I = [a, b]$ *be an interval of real numbers and assume*

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties:

1. $f(x, y)$ *is nondecreasing in* x *for each* $y \in [a, b]$ *and* $f(x, y)$ *is nonincreasing in* y *for each* $x \in [a, b]$.
2. *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$\begin{aligned} m &= f(m, M) \\ M &= f(M, m) \end{aligned}$$

   then $m = M$.

Then Eq.(2.9) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(2.9) converges to* $\overline{x}$.

Proof. Set

$$m_0 = a \textit{ and } M_0 = b$$

for each $i = 1, 2, 3, \cdots$

$$m_i = f(m_{i-1}, M_{i-1}) \textit{ and } M_i = f(M_{i-1}, m_{i-1})$$

then

$$m_1 = f(m_0, M_0) \geq a = m_0 \textit{ and } M_1 = f(M_0, m_0) \leq b = M_0$$

and

$$m_2 = f(m_1, M_1) \geq f(m_0, M_0) = m_1 \geq m_0$$
$$M_2 = f(M_1, m_1) \leq f(M_0, m_0) = M_1 \leq M_0$$

by induction, we have

$$m_0 \leq m_1 \cdots m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0$$

also

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(M_0, m_0) = M_1$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(m_0, M_0) = m_1$$

and

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(M_1, m_1) = M_2$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(m_1, M_1) = m_2$$

by induction, we have

$$m_i \leq x_n \leq M_i, \quad n \geq (i-1)k + i$$

set

$$m = \lim_{i \to \infty} m_i \textit{ and } M = \lim_{i \to \infty} M_i$$

then we have

$$m \leq \lim_{i \to \infty} \inf x_i \leq \lim_{i \to \infty} \sup x_i \leq M$$

by continuity of f

$$m = f(m, M) \textit{ and } M = f(M, m)$$

therefore in view of (2)

$$m = M = \overline{x}$$

□

Theorem 2.17. *[14] Let $I = [a, b]$ be an interval of real numbers and assume*

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties:

1. *$f(x, y)$ is non increasing in each of its arguments;*
2. *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$\begin{aligned} m &= f(M, M) \\ M &= f(m, m) \end{aligned}$$

   then $m = M$.

Then Eq.(2.9) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(2.9) converges to $\overline{x}$.

Proof. Set

$$m_0 = a \textit{ and } M_0 = b$$

for each $i = 1, 2, 3, \cdots$. Set

$$M_i = f(m_{i-1}, m_{i-1}) \textit{ and } m_i = f(M_{i-1}, M_{i-1})$$

Then

$$m_1 = f(M_0, M_0) \geq a = m_0, \textit{ and } M_1 = f(m_0, m_0) \leq b = M_0$$

and

$$m_2 = f(M_1, M_1) \geq f(M_0, M_0) = m_1 \geq m_0$$

$$M_2 = f(m_1, m_1) \leq f(m_0, m_0) = M_1 \leq M_0$$

By induction, we have

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0$$

Also

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(m_0, m_0) = M_1$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(M_0, M_0) = m_1$$

and

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(m_1, m_1) = M_2$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(M_1, M_1) = m_2$$

By induction, we have

$$m_i \leq x_n \leq M_i,\ for\ n \geq (i-1)k + i$$

Set

$$m = \lim_{i\to\infty} m_i \ and\ M = \lim_{i\to\infty} M_i$$

then clearly

$$M \geq \limsup_{i\to\infty} x_i \geq \liminf_{i\to\infty} x_i \geq m$$

and by the continuity of f,

$$m = f(M, M) \ and\ M = f(m, m)$$

therefore in view of (2)

$$m = M = \overline{x}$$

□

Theorem 2.18. *[14] consider the difference Eq.(2.9). Let $I = [a, b]$ be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties:

1. *$f(x, y)$ is non increasing in x for each $y \in [a, b]$, and $f(x, y)$ is non increasing in y for each $x \in [a, b]$.*

2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, M)$$
$$M = f(m, m)$$

then $m = M$.

3. The equation $f(x, y) = x$ has a unique positive solution.

Then Eq.(2.9)has a unique positive solution and every positive solution of Eq.(2.9) converges to $\overline{x}$.

Proof. set $m_0 = a$ and $M_0 = b$. for $i = 1, 2, 3, \cdots$

$$m_i = f(M_{i-1}, M_{i-1}) \text{ and } M_i = f(m_{i-1}, m_{i-1})$$

Then

$$m_1 = f(M_0, M_0) \geq a = m_0, \text{ and } M_1 = f(m_0, m_0) \leq b = M_0$$

and

$$m_2 = f(M_1, M_1) \geq f(M_0, M_0) = m_1 \geq m_0$$
$$M_2 = f(m_1, m_1) \leq f(m_0, m_0) = M_1 \leq M_0$$

By induction, we have

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0$$

Also

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(m_0, m_0) = M_1$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(M_0, M_0) = m_1$$

and

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(m_1, m_1) = M_2$$
$$x_{n+1} = f(x_n, x_{n-k}) \geq f(M_1, M_1) = m_2$$

By induction, we have

$$m_i \leq x_n \leq M_i, \quad n \geq (i-1)k + i$$

set

$$m = \lim_{i\to\infty} m_i \text{ and } M = \lim_{i\to\infty} M_i$$

then we have

$$m \le \lim_{i\to\infty} \inf x_i \le \lim_{i\to\infty} \sup x_i \le M$$

By continuity of f

$$m = f(M, M) \text{ and } M = f(m, m)$$

by assumption (2)

$$m = M = \overline{x}$$

□

Theorem 2.19. *[14] Assume that $f \in C\ [(0,\infty) \times (0,\infty), (0,\infty)]$ and that $f(x, y)$ is decreasing in both arguments. Let $\overline{x}$ be a positive equilibrium of equation Eq.(2.9), then every oscillatory solution of Eq.(2.9) has semicycle of length at most $k + 1$.*

Proof. When $k = 1$, the proof is presented as *theorem* 1.7.2 in [1]. We just give the proof of the theorem for $k = 2$. The other cases for $k \ge 3$ are similar and can be omitted. Assume that $\{x_n\}$ is an oscillatory solution with three consecutive terms x_{N-1}, x_N, x_{N+1} in a positive semicycle

$$x_{N-1} \ \ge \overline{x},\ x_N \ge \overline{x},\ x_{N+1} \ge \overline{x}$$

with at least one of the inequalities being strict. The proof in the case of negative semicycle is similar and is omitted.

Then by using the decreasing character of f. We obtain

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < f(\overline{x}, \overline{x}) = \overline{x}$$

which completes the proof.

For $k = 3$, assume that $\{x_n\}$ is an oscillatory solution with four consecutive terms $x_{N-1}, x_N, x_{N+1}, x_{N+2}$ in a negative semicycle

$$x_{N-1} \le \overline{x},\ x_N \ \le \overline{x},\ x_{N+1} \le \overline{x},\ x_{N+2} \le \overline{x}$$

with at least one of the inequalities being strict. The proof in the case of positive semicycle is similar and is omitted. Then by using the decreasing character of f. We obtain

$$x_{N+3} = f(x_{N+2}, x_{N-1}) > f(\overline{x}, \overline{x}) = \overline{x}$$

which completes the proof.

□

Theorem 2.20. *[14] Assume that* $f \in C\ [(0,\infty) \times (0,\infty), (0,\infty)]$ *is such that* $f(x,y)$ *is increasing in* x *for each fixed* y*.and* $f(x,y)$ *is decreasing in* y *for each fixed* x*. Let* $\overline{x}$ *be a positive equilibrium of equation Eq.(2.9) then every oscillatory solution of equation (2.9) has semicycle of length at least* $k+1$*.*

Proof. When $k = 1$, the proof is presented as $theorem 1.7.4$ in [14]. We just give the proof of the theorem for $k = 2$.the other cases for $k \geq 3$ are similar and can be omitted.

Assume that $\{x_n\}$ is an oscillatory solution with *three* consecutive terms

$$x_{N-1}, x_N, x_{N+1}$$

such that

$$x_{N-1} < \overline{x} < x_{N+1}$$

or

$$x_{N-1} > \overline{x} > x_{N+1}$$

we will assume that

$$x_{N-1} < \overline{x} < x_{N+1}$$

the other case is similar and will be omitted. Then by using decreasing character of f we obtain

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\overline{x}, \overline{x})$$

Now, if $x_N \geq \overline{x}$ then the result follows. Otherwise $x_N < \overline{x}$. Hence

$$x_{N+3} = f(x_{N+2}, x_N) > f(\overline{x}, \overline{x}) = \overline{x}$$

which shows that it has at least *three* terms in the positive semicycle

□

Theorem 2.21. *[14] Consider the difference equation*

$$x_{n+1} = f_0(x_n, x_{n-1})x_n + f_1(x_n, x_{n-1})x_{n-1}, \ n = 0, 1, 2, \cdots \tag{2.14}$$

with nonnegative initial conditions and

$$f_0, f_1 \in C[[0, \infty) \times [0, \infty), [0, 1)]$$

. Assume that the following hypothesis hold:

1. f_0 *and* f_1 *are non increasing in each of their arguments;*

2. $f_0(x, x) > 0$ *for all* $x \geq 0$*;*

3. $f_0(x, y) + f_1(x, y) < 1$ *for all* $x, y \in (0, \infty)$*.*
   Then the zero equilibrium of Eq.(2.14) is globally asymptotically stable.

Theorem 2.22. *[14] Assume that*

1. $f \in C\ [(0, \infty) \times (0, \infty), (0, \infty)]$*;*

2. $f(x, y)$ *is non increasing in* x *and decreasing in* y*;*

3. $xf(x, x)$ *is increasing in* x*;*

4. *The equation*

$$x_{n+1} = x_n f(x_n, x_{n-1}), \ n = 0, 1, 2, \cdots$$

   has a unique positive equilibrium $\overline{x}$*.*
   Then $\overline{x}$ *is globally asymptotically stable.*

Chapter 3

Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-k}}$

3 Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-k}}$

In this chapter we present the main part of this thesis, that is studying and investigating the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, n = 0, 1, 2, ... \qquad (3.1)$$

where the parameters α, β, A, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$ are non-negative real numbers with the solution is defined and $k \in \{1, 2, 3, ...\}$.

Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of Eq.(3.1).

It is worth mentioning that the results in [15] are special case of our main results. Where the global stability of Eq.(3.1) for $k = 1$ has been investigated in it. They showed that in respect to variation of the parameters, the positive equilibrium point is globally asymptotically stable or every solution lies eventually in an invariant interval.

Dehghan in [5] investigated the global stability, invariant intervals, the character of semi-cycles, and boundedness of the equation

$$x_{n+1} = \frac{x_n + p}{Bx_n + qx_{n-k}}, n = 0, 1, 2, ...$$

where the parameters p and q and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$ are positive real numbers, $k \in \{1, 2, 3, ...\}$.

3.1 Change of variables

Theorem 3.1. *The change of variable*

$$x_n = \frac{A}{B} y_n$$

reduces Eq.(3.1) to the difference equation

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-k}}, n = 0, 1, 2, 3, ... \tag{3.2}$$

where

$$p = \frac{\alpha B}{A^2}, q = \frac{\beta}{A}, r = \frac{C}{B}$$

with

$$p, q, r \in (0, \infty)$$

and

$$y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_0 \in (0, \infty)$$

Proof. Let

$$x_n = \frac{A}{B} y_n$$

then

$$x_{n+1} = \frac{A}{B} y_{n+1}$$

and

$$x_{n-k} = \frac{A}{B} y_{n-k}$$

Substitute in the Eq.(3.1), we get

$$\frac{A}{B} y_{n+1} = \frac{\alpha + \beta \frac{A}{B} y_n}{A + B \frac{A}{B} y_n + C \frac{A}{B} y_{n-k}}$$

by pulling a common factor $\frac{A}{B}$,

$$\frac{A}{B} y_{n+1} = \frac{\frac{A}{B}\left(\frac{B}{A}\alpha \ + \ \beta \ y_n\right)}{\frac{A}{B}\left(B + B \ y_n \ + \ C \ \ y_{n-k}\right)}$$

hence

$$y_{n+1} = \frac{\frac{B}{A}\left(\frac{B}{A}\alpha + \beta y_n\right)}{B\left(1 + y_n + \frac{C}{B} \ y_{n-k}\right)}$$

then

$$y_{n+1} = \frac{\frac{B\alpha}{A^2} + \frac{\beta}{A} y_n}{1 + y_n + \frac{C}{B}\ y_{n-k}}$$

Now let

$$p = \frac{\alpha B}{A^2}, q = \frac{\beta}{A}, r = \frac{C}{B}$$

reduces Eq.(3.1) to

$$y_{n+1} = \frac{p + q y_n}{1 + y_n + r y_{n-k}}$$

Thus, the proof has been completed. □

Definition 3.2. Let I be some interval of real numbers and let

$$f : I_{k+1} \longrightarrow I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \cdots x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}),\ n = 0, 1, \cdots \tag{3.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 3.3. The solution $\{y_n\}_{n=-k}^{\infty}$ of the difference equation $y_{n+1} = f(y_n, y_{n-1}, \cdots, y_{n-k})$ is periodic if there exists a positive integer p such that $y_{n+p} = y_n$. The smallest such integer p is called the prime period of the solution of the difference equation.

Definition 3.4. The equilibrium point $\overline{y}$ of the equation

$$y_{n+1} = f(y_n, y_{n-1}, \cdots, y_{n-k}), n = 0, 1, \cdots$$

is the point that satisfies the condition

$$\overline{y} = f(\overline{y}, \overline{y}, \cdots, \overline{y})$$

.

3.2 Equilibrium Points

In this section we find the unique positive equilibrium point of the nonlinear difference equation

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-k}}, \ n = 0, 1, \cdots \tag{3.4}$$

where the parameters p, q, r and the initial conditions $y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_0$ are positive real numbers, and $k \in \{1, 2, \cdots\}$.

To find the equilibrium point, we solve the following equation

$$\overline{y} = \frac{p + q\overline{y}}{1 + \overline{y} + r\overline{y}}$$

hence

$$\overline{y}\,(1 + \overline{y} + r\overline{y}) = p + q\overline{y}$$

by rearranging the terms, we get:

$$(1 + r)\,\overline{y}^2 + (1 - q)\,\overline{y} - p = 0$$

Solving this quadratic equation, we get the equilibrium points

$$\overline{y} = \frac{(q - 1) \pm \sqrt{(q - 1)^2 + 4p(r + 1)}}{2(r + 1)}$$

The only positive solution is:

$$\overline{y} = \frac{(q - 1) + \sqrt{(q - 1)^2 + 4p(r + 1)}}{2(r + 1)}$$

3.3 Linearized equation

Let $f(x, y)$ have a continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$. Let h and k be increments small enough to put the point $S(a + h, b + k)$ and the line segment joining it to P inside R. We parametrize the segment PS as

$$x = a + th, \ y = b + tk, \ 0 \leq t \leq 1.$$

If $F(t) = f(a+th, b+tk)$, the Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable, F' is a differentiable function of t and

$$F'' = \frac{\partial F'}{\partial x}\frac{dx}{dt} + \frac{\partial F'}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{dx}(hf_x + kf_y).h + \frac{\partial}{dy}(hf_x + kf_y).k$$

$$= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}.$$

Since F and F' are continuous on $[0,1]$ and F' is differentiable on $(0,1)$, we can apply Taylor's formula with $n = 2$ and $a = 0$ to obtain

$$F(1) = F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2}$$

$$F(1) = F(0) + F'(0) + \frac{1}{2}F''(c) \tag{3.5}$$

for some c between 0 and 1. writing Eq.(3.5) in terms of f gives

$$f(a+h, b+k) = f(a,b)+hf_x(a,b)+kf_y(a,b)+\frac{1}{2}(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy})|_{(a+ch,b+ck)} \tag{3.6}$$

Now substitute $\overline{x}$ and $\overline{x}$ for a and b, and $x-\overline{x}$ and $y-\overline{x}$ for h and k respectively in Eq.(3.6), and rearrange the result as

$$f(x,y) = f(\overline{x},\overline{x})+f_x(\overline{x},\overline{x})(x-\overline{x})+f_y(\overline{x},\overline{x})(y-\overline{x})+\frac{1}{2}((x-\overline{x})^2 f_{xx}+2(x-\overline{x})(y-\overline{x})f_{xy}++(y-\overline{x})^2 f_{yy}) \tag{3.7}$$

where

$$f(\overline{x},\overline{x}) + f_x(\overline{x},\overline{x})(x-\overline{x}) + f_y(\overline{x},\overline{x})(y-\overline{x})$$

is called the Linearization term $L(x,y)$, and

$$\frac{1}{2}((x-\overline{x})^2 f_{xx} + 2(x-\overline{x})(y-\overline{x})f_{xy} + (y-\overline{x})^2 f_{yy})|_{(\overline{x}+c(x-\overline{x}),\overline{x}+c(y-\overline{x})))}$$

is the error term $E(x,y)$.

In studying the behavior of the local stability of Eq.(3.7), its enough to study the homogeneous part

$$f(x,y) = f_x(\overline{x},\overline{x})x + f_y(\overline{x},\overline{x})y \tag{3.8}$$

Which is called the scalar form.

In matrix form $Z = AX$, where $A = \begin{pmatrix} f_x & 0 \\ 0 & f_y \end{pmatrix}$ and

$X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Now let $x = x_n$, $y = x_{n-k}$, and $x_{n+1} = f(x,y)$ then Eq.(3.8) becomes

$$x_{n+1} = f(x_n, x_{n-k}) = f_x(\overline{x},\overline{x})x_n + f_y(\overline{x},\overline{x})x_{n-k}.$$

To find the linearized equation of our problem, consider

$$f(x,y) = \frac{p+qx}{1+x+ry}$$

then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{q(1+x+ry)-(p+qx)}{(1+x+ry)^2} \\ &= \frac{q+x+qry-p-qx}{(1+x+ry)^2} \\ &= \frac{q-p+qry}{(1+x+ry)^2} \end{aligned}$$

which implies that

$$\begin{aligned} \frac{\partial f}{\partial x}(\overline{y},\overline{y}) &= \frac{q-p+qr\overline{y}}{(1+\overline{y}+r\overline{y})^2} \\ &= \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2} \end{aligned}$$

similarly

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{0(1+x+ry)-r(p+qx)}{(1+x+ry)^2}\\ &= \frac{-r(p+qx)}{(1+x+ry)^2}\end{aligned}$$

thus

$$\frac{\partial f}{\partial y}(\overline{y},\overline{y}) = \frac{-r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}$$

so the linearized equation which is associated to Eq.(3.4) about the equilibrium point $\overline{y}$ is :

$$z_{n+1} = \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}z_n - \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}z_{n-k}$$

i.e

$$z_{n+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}z_n + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}z_{n-k} = 0 \tag{3.9}$$

and its characteristic equation is:

$$\lambda^{n+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}\lambda^n + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}\lambda^{n-k} = 0 \tag{3.10}$$

which implies

$$\lambda^{k+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}\lambda^k + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2} = 0 \tag{3.11}$$

3.4 The Local Stability

The following two lemmas are important for the study of local stability.

Lemma 3.5. *[11] [13] Assume that $a, b \in \Re$ and $k \in \{1, 2, 3, \cdots\}$. Then a necessary and sufficient condition for asymptotic stability of the equation*

$$x_{n+1} + ax_n + bx_{n-k} = 0, n = 0, 1, 2, \cdots$$

is that

$$|a| < 1 + b < 2$$

Lemma 3.6. *[14] [11] Assume that all the roots of the characteristic equation of the above equation lie inside the unit circle, then the positive equilibrium point is locally asymptotically stable.*

Theorem 3.7. *The unique positive equilibrium point of Eq.(3.4) is locally asymptotically stable for all values of the parameters p, q, and r provided that all roots of Eq.(3.11) lie inside the unit circle.*

3.5 Boundedness of Solutions

Theorem 3.8. *Every solution of Eq.(3.4) is bounded from above and from below by a positive constant.*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.4), so clearly if the solution is bounded from above by a constant M, then

$$y_{n+1} \geq \frac{p}{1+(1+r)M}$$

and so its bounded from below.
Now assume for the sake of contradiction that the solution is not bounded from above, then there exists a subsequence $\{y_{n_m+1}\}_{m=0}^{\infty}$ such that

$\lim_{m\to\infty} y_{n_m} = \infty$, $\lim_{m\to\infty} y_{n_m+1} = \infty$ and $y_{n_m+1} = \max\{y_n : n \leq n_m\}$ for $m \geq 0$.

so for Eq.(3.4), we see that

$$y_{n+1} < p + qy_n, \text{ for } n \geq 0$$

and so

$$\lim_{m\to\infty} y_{n_m} = \lim_{m\to\infty} y_{n_m-1} = \infty$$

hence, for sufficiently large m

$$0 < y_{n_m+1} - y_{n_m} = \frac{p + [(q-1) - y_{n_m} - ry_{n_m-k}]y_{n_m}}{1 + y_{n_m} + ry_{n_m-k}} < 0$$

which is a contradiction.

□

3.6 Invariant Interval

Definition 3.9. Invariant Interval of the difference equation Eq.(3.3) is an interval with the property that if $k+1$ consecutive terms of the solution fall in I then all subsequent terms of the solution also belong to I. In other words, I is an invariant interval for Eq.(3.3) if $y_{N-k+1}, \cdots, y_{N-1}, y_N \in I$ for some $N \geq 0$, then $y_n \in I$ for every $n > N$.

Theorem 3.10. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.4). Then the following statements are true:*

1. *Suppose $p \leq q$ and assume that for some $N \geq 0$,*
$$y_{N-k},\ y_{N-k+1},\ \cdots, y_N \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$$
then $y_n \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$ for all $n > N$.

2. *Suppose $q < p < q(rq+1)$ and assume that for some $N \geq 0$,*
$$y_{N-k},\ y_{N-k+1},\ \cdots, y_N \in [\frac{p-q}{qr}, q]$$
then $y_n \in [\frac{p-q}{qr}, q]$ for all $n > N$.

3. *Suppose $p > q(rq+1)$ and assume that for some $N \geq 0$,*
$$y_{N-k},\ y_{N-k+1},\ \cdots, y_N \in [q, \frac{p-q}{qr}]$$
then $y_n \in [q, \frac{p-q}{qr}]$ for all $n > N$.

Proof. 1. Set
$$g(x) = \frac{p+qx}{1+x} \text{ and } b = \frac{(q-1)+\sqrt{(q-1)^2+4p}}{2}$$
and observe that g is an increasing function and $g(b) = b$, using Eq.(3.4), we see that when $y_{N-k},\ y_{N-k+1},\ \cdots, y_N \in [0, b]$, then
$$y_{N+1} = \frac{p+qy_N}{1+y_N+ry_{N-k}} \leq \frac{p+qy_N}{1+y_N} = g(y_N) \leq b$$
The proof follows by induction.

2. Take the function
$$f(x,y)=\frac{p+qx}{1+x+ry}$$
its clear that this function is increasing in x for $y>\frac{p-q}{qr}$. Using Eq.(3.4),

we see that if $y_{N-k},\ y_{N-k+1},\ \cdots,y_N\in[\frac{p-q}{qr},q]$,

then
$$y_{N+1}=\frac{p+qy_N}{1+y_N+ry_{N-k}}=f(y_N,y_{N-k})\leq f(q,\frac{p-q}{qr})=q$$
and by using the condition $p<q(rq+1)$, we obtain
$$y_{N+1}=\frac{p+qy_N}{1+y_N+ry_{N-k}}\quad=f(y_N,y_{N-k})\geq f(\frac{p-q}{qr},q)=\frac{q(pr+p-q)}{(rq)^2+rq+p-q}>\frac{p-q}{qr}.$$
and the proof follows by induction.

3. Take the function
$$f(x,y)=\frac{p+qx}{1+x+ry}$$
its clear that this function is decreasing in x for $y<\frac{p-q}{qr}$. Using Eq.(3.4), we see that for $y_{N-k},\ y_{N-k+1},\ \cdots,y_N\in[q,\frac{p-q}{qr}]$, then
$$y_{N+1}=\frac{p+qy_N}{1+y_N+ry_{N-k}}=f(y_N,y_{N-k})\geq f(\frac{p-q}{qr},\frac{p-q}{qr})=q$$
and by using the condition $p>q(rq+1)$, we obtain
$$y_{N+1}=\frac{p+qy_N}{1+y_N+ry_{N-k}}=f(y_N,y_{N-k})\leq f(q,q)=\frac{p+q^2}{1+(r+1)q}<\frac{p-q}{qr}$$
The proof follows by induction.

$\square$

3.7 Existence of Two cycles

Definition 3.11. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.4). We say that the solution has a prime period *two* if the solution eventually takes the form:

$$\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots$$

where $\phi,\ \psi$ are distinct and positive.

Theorem 3.12. *If k is even, then Eq.(3.4) has no nonnegative distinct prime period two solution.*

Proof. Let k even, assume for the sake of contradiction that there is distinct nonnegative real numbers $\phi,\ \psi$ such that

$$\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots$$

is prime period two solution of Eq.(3.4), then ϕ, ψ satisfy :

$$\psi = \frac{p + q\phi}{1 + \phi + r\phi}$$

and

$$\phi = \frac{p + q\psi}{1 + \psi + r\psi}$$

then by substituting ϕ into the equation of ψ, we get easily by a simple calculation that

$$(1 + r + q)\psi^2 + (1 - q^2)\psi - (p + qp) = 0$$

solving this quadratic equation for ψ, we get

$$\psi = \frac{(q^2 - 1) \pm \sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)}}{2(1 + r + q)}$$

but

$$\sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)} > (q^2 - 1)$$

and ψ is nonnegative, then

$$\psi = \frac{(q^2-1)+\sqrt{(q^2-1)^2+4(p+qp)(1+r+q)}}{2(1+r+q)}$$

Now again the same steps for ϕ, substituting ψ into ϕ, we get that

$$\phi = \frac{(q^2-1)+\sqrt{(q^2-1)^2+4(p+qp)(1+r+q)}}{2(1+r+q)}$$

which implies

$$\psi = \phi$$

which contradicts the hypothesis that ψ and ϕ are distinct nonnegative real numbers.

□

Theorem 3.13. *If k is odd, then Eq.(3.4), has no nonnegative distinct prime period two solution.*

Proof. Let k be odd, and assume that for the sake of contradiction that there is distinct nonnegative real numbers ϕ and ψ such that

$$\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots$$

is prime period two solution of Eq.(3.4), then ϕ, ψ satisfy :

$$\phi = \frac{p+q\psi}{1+\psi+r\phi}$$

and

$$\psi = \frac{p+q\phi}{1+\phi+r\psi}$$

by multiplying, we get

$$\phi+\phi\psi+r\phi^2 = p+q\psi$$

$$\psi + \phi\psi + r\psi^2 = p + q\phi$$

by rearranging the above equation by some algebra we get:

$$q(\psi - \phi) + (\psi - \phi) + r(\psi^2 - \phi^2) = 0$$
$$q(\psi - \phi) + (\psi - \phi) + r(\psi - \phi)(\psi + \phi) = 0$$

we can divide the above equation by $(\psi - \phi)$, since $\phi \neq \psi$, then

$$q + 1 + r(\psi + \phi) = 0$$

which implies that

$$\psi + \phi = \frac{-q-1}{r}$$

which is a contradiction for that ψ and ϕ are both nonnegative.

□

Corollary 3.14. *Eq.(3.4) posses no prime period two solution.*

3.8 Analysis of Semicycles and Oscillation

Analysis of semicycles of the solution of Eq.(3.4) is a powerful tool for a detailed understanding of the entire character of solutions.

Definition 3.15. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.3) and $\overline{x}$ be a positive equilibrium point. We now give the definitions of positive and negative semicycle of a solution of Eq.(3.3) relative to the equilibrium point $\overline{x}$

- A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(3.3) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ such that

  $$either\ l = -1,\ or\ l < -1\ and\ x_{l-1} < \overline{x}$$

  and

  $$either\ m = \infty,\ or\ m < \infty\ and\ x_{m+1} < \overline{x}$$

- A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(3.3) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\textit{either } l = -1, \textit{ or } l < -1 \textit{ and } x_{l-1} \geq \overline{x}$$

and

$$\textit{either } m = \infty, \textit{ or } m < \infty \textit{ and } x_{m+1} \geq \overline{x}$$

Definition 3.16. (Oscillation)

1. A sequence $\{x_n\}$ is said to **oscillate a bout zero** or simply to **oscillate** if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence is called **strictly oscillatory** if for n_0, there exist $n_1, n_2 \geq n_0$ such that $x_{n_1} x_{n_2} < 0$.

2. A sequence x_n is said to **oscillate about** $\overline{x}$ if the sequence $x_n - \overline{x}$ oscillates. The sequence x_n is called **strictly oscillatory about** $\overline{x}$ if the sequence $x_n - \overline{x}$ is strictly oscillatory.

Analysis of Semicycles Based on Invariant Intervals

The aim of this part is to present the analysis of semicycles of solution of Eq.(3.4) relative to equilibrium point $\overline{y}$ and based on invariant interval of Eq.(3.4).

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.4). Then observe that the following identities are true:

$$y_{n+1} - q = (qr)[\frac{\frac{p-q}{qr} - y_{n-k}}{1 + y_n + ry_{n-k}}] \tag{3.12}$$

$$y_{n+1} - (\frac{p-q}{qr}) = \frac{qr(q - \frac{p-q}{qr})y_n + qr(y_{n-k} - \frac{p-q}{qr}) + pr(q - y_{n-k})}{qr(1 + y_n + ry_{n-k})} \tag{3.13}$$

$$y_n - y_{n+4} = \frac{M(y_n - q)}{(1 + y_{n+3})(1 + y_{n+1} + ry_n) + r(p + qy_{n+1})} + qr(y_n - \frac{p-q}{qr})y_{n+1} + y_n + ry_n^2 - p \tag{3.14}$$

$$y_{n+1} - \overline{y} = \frac{(\overline{y} - q)(\overline{y} - y_n) + \overline{y}r(\overline{y} - y_{n-k})}{1 + y_n + ry_{n-k}} \tag{3.15}$$

where

$$M = y_{n+1}y_{n+3} + y_{n+3} + y_{n+1} + ry_ny_{n+3}$$

so the proof of the following lemmas are straight forward consequence of the above identities.

Lemma 3.17. *Suppose that $p > q(qr+1)$ and let $\{y_n\}_{n=-k}^{\infty}$ be solutions of Eq.(3.4), then the following statements are true:*

1. *If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then $y_{N+1} > q$.*

2. *If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} = q$.*

3. *If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} < q$.*

4. *If for some $N \geq 0$, $q < y_{N-k} < \frac{p-q}{qr}$. Then $q < y_{N+1} < \frac{p-q}{qr}$.*

5. *If for some $N \geq 0$, $q < y_{N-k}, \cdots, y_{N-1}, y_N < \frac{p-q}{qr}$. Then $q < y_n < \frac{p-q}{qr}$. That is $[q, \frac{p-q}{qr}]$ is an invariant interval for Eq.(3.4).*

6. *If for some $N \geq 0$, $\overline{y} < y_{N-k}$, and $\overline{y} < y_N$. Then $y_{N+1} < \overline{y}$.*

7. *If for some $N \geq 0$, $\overline{y} \geq y_{N-k}$ and $\overline{y} \geq y_N$. Then $y_{N+1} \geq \overline{y}$.*

8. *$q < \overline{y} < \frac{p-q}{qr}$.*

Proof. 1. If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then we can conclude that $y_{N+1} - q > 0$ using Eq.(3.12). Which implies that $y_{N+1} > q$.

2. If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} - q = 0$ using Eq.(3.12), which implies that $y_{N+1} = q$.

3. If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} - q < 0$ using Eq.(3.12), which implies that $y_{N+1} < q$.

4. If for some $N \geq 0$, $q < y_{N-k} < \frac{p-q}{qr}$, we can see that if $y_{N-k} < \frac{p-q}{qr}$. Then $y_{N+1} > q$ by (1). Similarly if $y_{N-k} > q$, then $y_{N+1} - \frac{p-q}{qr} < 0$ which implies that $y_{N+1} < \frac{p-q}{qr}$ using Eq.(3.13). Then we conclude that $q < y_{N+1} < \frac{p-q}{qr}$.

5. We see in (4), that If for some $N \geq 0$, $q < y_{N-k} < \frac{p-q}{qr}$. Then $q < y_{N+1} < \frac{p-q}{qr}$. Now we can see that if $q < y_{N-k}, ..., y_{N-1}, y_N < \frac{p-q}{qr}$, then $y_{N+1}, y_{N+2}, ... \in (q, \frac{p-q}{qr})$ using Eq.(3.13), which implies that $q < y_n < \frac{p-q}{qr}$. That is $[q, \frac{p-q}{qr}]$ is invariant interval for Eq.(3.4).

6. If for some $N \geq 0$, $\overline{y} < y_{N-k}$, and $\overline{y} < y_N$. Then using Eq.(3.15), we see that $y_{N+1} - \overline{y} < 0$, i.e, $y_{N+1} < \overline{y}$.

7. If for some $N \geq 0$, $\overline{y} \geq y_{N-k}$, and $\overline{y} \geq y_N$. Then using Eq.(3.15), we see that $y_{N+1} - \overline{y} \geq 0$, i.e, $y_{N+1} \geq \overline{y}$.

8. Using (5), since $[q, \frac{p-q}{qr}]$ is invariant interval, then $q < \overline{y} < \frac{p-q}{qr}$.

□

The following theorem is the extension of theorem(2.19).

Corollary 3.18. *Every nontrivial and oscillatory solution of Eq.(3.4) which lies in the interval $[q, \frac{p-q}{qr}]$ oscillates about the equilibrium point $\overline{y}$, with semicycle of length at most $k+1$.*

Proof. Corollary(3.18) is extension of Theorem(**??**), where Eq.(3.4) is decreasing in both arguments in $[q, \frac{p-q}{qr}]$. Then every oscillatory solution of Eq.(3.4) has a semicycle of length at most $k+1$. □

Lemma 3.19. *Suppose that $q < p < q(qr+1)$ and let $\{y_n\}_{n=-k}^{\infty}$ be solution of equation (3.4), then the following statements are true:*

1. *If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then $y_{N+1} > q$.*

2. *If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} = q$.*

3. *If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} < q$.*

4. *If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k} < q$. Then $\frac{p-q}{qr} < y_{N+1} < q$.*

5. *If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k}, \cdots, y_{N-1}, y_N < q$. Then $\frac{p-q}{qr} < y_n < q$. That is $[\frac{p-q}{qr}, q]$ is an invariant interval for Eq.(3.4).*

6. *$\frac{p-q}{qr} < \overline{y} < q$.*

Proof.

1. If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then we can conclude that $y_{N+1} - q > 0$ using Eq.(3.12). Which implies that $y_{N+1} > q$.

2. If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} - q = 0$ using Eq.(3.12), which implies that $y_{N+1} = q$.

3. If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} - q < 0$ using Eq.(3.12), which implies that $y_{N+1} < q$.

4. If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k} < q$, we can see that if $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} < q$ by (1). Similarly if $y_{N-k} < q$, then $y_{N+1} - \frac{p-q}{qr} > 0$ which implies that $y_{N+1} > \frac{p-q}{qr}$ using Eq.(3.13). Then we conclude that $\frac{p-q}{qr} < y_{N+1} < q$.

5. We see in (4), that If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k} < q$. Then $\frac{p-q}{qr} < y_{N+1} < q$. Now we can see that if $\frac{p-q}{qr} < y_{N-k}, ..., y_{N-1}, y_N < q$, then $y_{N+1}, y_{N+2}, ... \in (\frac{p-q}{qr}, q)$ using Eq.(3.13), which implies that $\frac{p-q}{qr} < y_n < q$. That is $[\frac{p-q}{qr}, q]$ is invariant interval for Eq.(3.4).

6. Using (5), since $[q, \frac{p-q}{qr}]$ is invariant interval, then $\frac{p-q}{qr} < \overline{y} < q$.

□

The following theorem is extension of theorem(2.20).

Corollary 3.20. *Every nontrivial and oscillatory solution* $\{y_n\}_{n=-k}^{\infty}$ *of Eq.(3.4) which lies in the interval* $[\frac{p-q}{qr}, q]$ *oscillates about the equilibrium point* $\overline{y}$*, with semicycle of length at least* $k+1$*.*

Proof. Corollary(3.20) is extension of Theorem(**??**), where Eq.(3.4) is increasing in x for each fixed y, and decreasing in y for each fixed x in the interval $[\frac{p-q}{qr}, q]$. Then every oscillatory solution of Eq.(3.4) oscillates about the equilibrium point $\overline{y}$ with semicycle of length at least $k+1$. □

Finally: we will discuss thoroughly the analysis of semicycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that $p = q(qr+1)$.

In this case Eq.(3.4) reduces to the following

$$y_{n+1} = \frac{q + rq^2 + qy_n}{1 + y_n + ry_{n-k}}$$

and the equilibrium point is $\overline{y} = q$, then the identities(3.12) through (3.15) reduces $y_{n+1} - q$, to

$$y_{n+1} - q = \frac{qr}{1 + y_n + ry_{n-k}}(q - y_{n-k}) \tag{3.16}$$

Furthermore, if $qr < 1$, then

$$\lim_{n\to\infty} y_n = \overline{y}. \tag{3.17}$$

Remark

1. Identity (3.16) follows by straight forward computation.

2. Limit in (3.17) is a consequence of the fact that in this case $qr \in (0,1)$ and Eq.(3.4) has no prime period two solution.

Lemma 3.21. *Suppose that* $p = q(qr+1)$ *and let* $\{y_n\}_{n=-k}^{\infty}$ *be solutions of Eq.(3.4), then the following statements are true:*

1. *If for some* $N \geq 0$, $y_{N-k} < q$. *Then* $y_{N+1} > q$.

2. *If for some* $N \geq 0$, $y_{N-k} = q$. *Then* $y_{N+1} = q$.

3. *If for some* $N \geq 0$, $y_{N-k} > q$. *Then* $y_{N+1} < q$.

Proof.

1. If for some $N \geq 0$, $y_{N-k} < q$, then by substitute $p = q(qr+1)$ in Eq.(3.12), we get $y_{N+1} - q > 0$ which implies that $y_{N+1} > q$.

2. If for some $N \geq 0$, $y_{N-k} = q$, then by substitute $p = q(qr+1)$ in Eq.(3.12), we get $y_{N+1} - q = 0$ which implies that $y_{N+1} = q$.

3. If for some $N \geq 0$, $y_{N-k} > q$, then by substitute $p = q(qr+1)$ in Eq.(3.12), we get $y_{N+1} - q < 0$ which implies that $y_{N+1} < q$.

□

Theorem 3.22. *Suppose that* $p = q(qr+1)$ *and let* $\{y_n\}_{n=-k}^{\infty}$ *be a nontrivial solution of Eq.(3.4), then* $\{y_n\}_{n=-k}^{\infty}$ *oscillates about the equilibrium point* q.

Proof. We notice that by using lemma(3.21) if $y_{N-k} < q$ then $y_{N+1} > q$, and if $y_{N-k} > q$ then $y_{N+1} < q$, which means that the solution $\{y_n\}_{n=-k}^{\infty}$ oscillates about the equilibrium point q. □

Now assume that the solutions does not eventually lie in the invariant interval.

Assume that $p > q(qr+1)$, let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.4) which does not eventually lie in the interval $I = [q, \frac{p-q}{qr}]$, then it can be observed that the solution oscillates about the equilibrium point relative to $[q, \frac{p-q}{qr}]$ essentially in one of the following two ways:

- $k+1$ consecutive terms in $(\frac{p-q}{qr},\infty)$, are followed by $k+1$ consecutive terms in $(\frac{p-q}{qr},\infty)$, and so on. The solution never meets the interval $(q,\frac{p-q}{qr})$.

- There exists exactly k terms in $(\frac{p-q}{qr},\infty)$ which is followed by k terms in $(q,\frac{p-q}{qr})$ which is followed by k terms in $(0,1)$ which is followed by k terms in $(q,\frac{p-q}{qr})$ which is followed by k terms in $(\frac{p-q}{qr},\infty)$ and so on.
The solution meets consecutively the intervals :

$$\cdots,(\frac{p-q}{qr},\infty),(q,\frac{p-q}{qr}),(0,1),(q,\frac{p-q}{qr}),(\frac{p-q}{qr},\infty),\cdots$$

in order with k terms per interval.

The situation is essentially the same relative to the interval $(\frac{p-q}{qr},q)$, when $q<p<q(qr+1)$.
And the same thing is done when $p=q(qr+1)$.

3.9 Global Asymptotic stability

The next results are about the global stability for the positive equilibrium of Eq.(3.4).

Theorem 3.23. *:[1] Let $I = [a, b]$ be an interval of real numbers and assume*

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties

1. *$f(x, y)$is non increasing in each of it's arguments;*
2. *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$\begin{aligned} m &= f(M, M) \\ M &= f(m, m) \end{aligned}$$

   then $m = M$.

Then

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, \cdots \tag{3.18}$$

has a unique equilibrium $\overline{y} \in [a, b]$ and every solution of Eq.(3.18) converges to $\overline{y}$.

Theorem 3.24. *:[1] Let $I = [a, b]$ be an interval of real numbers and assume*

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuously function satisfying the following properties

1. *$f(x, y)$is non decreasing in x for each $y \in [a, b]$ and $f(x, y)$is non increasing in y for each $x \in [a, b]$*
2. *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(m, M)$$
$$M = f(M, m)$$

then $m = M$.

Then Eq.(3.18) has a unique equilibrium $\overline{y} \in [a, b]$ *and every solution of Eq.(3.18) converges to* $\overline{y}$.

Now we will apply these theorems on our equation.

Theorem 3.25. *Assume that* $p > q(qr + 1)$, *then the positive equilibrium of Eq.(3.4) on the interval* $[q, \frac{p-q}{qr}]$ *is globally asymptotically stable.*

Proof. this proof can easily done depending on theorem (3.23).
Assume that $p > q(qr + 1)$ and consider the function

$$f(x, y) = \frac{p + qx}{1 + x + ry}$$

First, note that $f(x, y)$ on the interval $[q, \frac{p-q}{qr}]$ is non increasing function in both of its arguments x, y.

Second, Let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \text{ and } f(M, M) = m$$

then

$$M = \frac{p + qm}{1 + m + rm}$$

and

$$m = \frac{p + qM}{1 + M + rM}$$

But we showed before that our equation has no periodic two solution, then the only solution is $m = M$.

Then both conditions of theorem(3.23) hold, therefore if $\overline{y}$ is an equilibrium point of Eq.(3.4), then every solution of Eq.(3.4) converges to $\overline{y}$ in the interval $[q, \frac{p-q}{qr}]$.
As $\overline{y}$ is asymptotically stable, then it is globally asymptotically stable on $[q, \frac{p-q}{qr}]$.

□

Theorem 3.26. *Assume that $q < p < q(qr+1)$, then the positive equilibrium of Eq.(3.4) on the interval $[\frac{p-q}{qr}, q]$ is globally asymptotically stable.*

Proof. This proof can be easily done depending on theorem (3.24).
Assume that $q < p < q(qr+1)$ and consider the function

$$f(x,y) = \frac{p+qx}{1+x+ry}$$

First, note that $f(x,y)$ on the interval $[\frac{p-q}{qr}, q]$ is nondecreasing function in x, and nonincreasing y.

Second, Let $(m, M) \in [a,b] \times [a,b]$ is a solution of the system

$$f(m,M) = m \text{ and } f(M,m) = M$$

then

$$m = \frac{p+qm}{1+m+rM}$$

and

$$M = \frac{p+qM}{1+M+rm}$$

But we showed before that our equation has no periodic two solution, then the only solution is $m = M$.

Then both conditions of theorem (3.24) hold, then if $\overline{y}$ is an equilibrium point of Eq.(3.4), then every solution of Eq.(3.4) converges to $\overline{y}$ in the Interval $[\frac{p-q}{qr}, q]$.
as $\overline{y}$ is asymptotically stable, then it is globally asymptotically stable on $[\frac{p-q}{qr}, q]$.

□

Theorem 3.27. *Assume that $p \leq q$, then the positive equilibrium of Eq.(3.4) on the interval $[0, \frac{(q-1)+\sqrt{(q-1)^2+4p}}{2}]$ is globally asymptotically stable.*

Proof. This proof can be easily done depending on theorem(3.24).
Assume that $p \leq q$ and consider the function

$$f(x, y) = \frac{p + qx}{1 + x + ry}$$

First, note that $f(x, y)$ on the interval $[0, \frac{(q-1)+\sqrt{(q-1)^2+4p}}{2}]$ is nondecreasing function in x, and nonincreasing y.

Second, Let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, M) = m \text{ and } f(M, m) = M$$

then

$$m = \frac{p + qm}{1 + m + rM}$$

and

$$M = \frac{p + qM}{1 + M + rm}$$

But we showed before that our equation has no periodic two solution, then the only solution is $m = M$.

$\square$

3.10 Numerical Discussion

In this section, we will study the global stability of our equation numerically based on some data and figures that we can get using MATLAB 6.5.

Example 3.1. *Assume Eq.(3.2) holds. take $k = 3$, $p = 4$, $r = 2$ and $q = 1$. So the equation will be*

$$y_{n+1} = \frac{4 + y_n}{1 + y_n + 2y_{n-3}}, \ n = 0, 1, 2, \cdots$$

We assumed that the initial points $\{x_0, x_1, x_2, x_3\}$ all to be $\in (0, \infty)$ and are $\{0.1, 0.2, 0.3, 0.4\}$.

The theoretical positive equilibrium point will be $\overline{y} = 1.154700538$.

Figure.1. shows the behavior of the equilibrium point of the $y_{n+1} = \frac{4+y_n}{1+y_n+2y_{n-3}}$. It shows that the equilibrium point $\overline{y}$ is globally asymptotically stable, as we have shown theoretically.

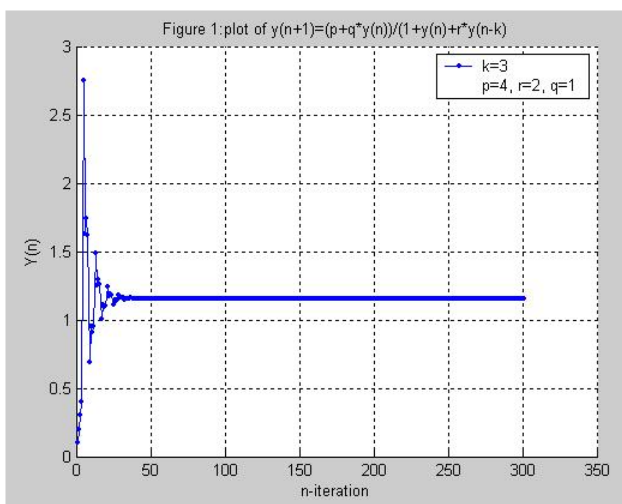


Table.1. shows that the numerically equilibrium point $\overline{y} = 1.1547$.

N	X(n)	N	X(n)	N	X(n)	N	X(n)
1	0.1000	26	1.1458	51	1.1544	276	1.1547
2	0.2000	27	1.1378	52	1.1545	277	1.1547
3	0.3000	28	1.1422	53	1.1552	278	1.1547
4	0.4000	29	1.1780	54	1.1547	279	1.1547
5	2.7500	30	1.1585	55	1.1549	280	1.1547
6	1.6265	31	1.1634	56	1.1548	281	1.1547
7	1.7438	32	1.1609	57	1.1544	282	1.1547
8	1.6208	33	1.1426	58	1.1547	283	1.1547
9	0.6921	34	1.1532	59	1.1546	284	1.1547
10	0.9488	35	1.1503	60	1.1547	285	1.1547
11	0.9103	36	1.1516	61	1.1548	286	1.1547
12	0.9531	37	1.1611	62	1.1547	287	1.1547
13	1.4841	38	1.1553	63	1.1547	288	1.1547
14	1.2516	39	1.1570	64	1.1547	289	1.1547
15	1.2896	40	1.1562	65	1.1546	290	1.1547
16	1.2607	41	1.1513	66	1.1547	291	1.1547
17	1.0061	42	1.1545	67	1.1547	292	1.1547
18	1.1102	43	1.1535	68	1.1547	293	1.1547
19	1.0897	44	1.1540	69	1.1547	294	1.1547
20	1.1038	45	1.1565	70	1.1547	295	1.1547
21	1.2400	46	1.1547	71	1.1547	296	1.1547
22	1.1748	47	1.1553	72	1.1547	297	1.1547
23	1.1885	48	1.1551	73	1.1547	298	1.1547
24	1.1803	49	1.1538	74	1.1547	299	1.1547
25	1.1116	50	1.1547	75	1.1547	300	1.1547

Table 1: Solution of DE $y_{n+1} = \frac{4+y_n}{1+y_n+2y_{n-3}}$

Example 3.2. *Assume Eq.(3.2) holds. take $k = 3$, $p = 3$, $r = 4$ and $q = 2$. So the equation will be*

$$y_{n+1} = \frac{3 + 2y_n}{1 + y_n + 4y_{n-3}}, \ n = 0, 1, 2, \cdots$$

We assumed that the initial points $\{x_0, x_1, x_2, x_3\}$ all to be $\in (0, \infty)$ and are $\{0.1, 0.2, 0.3, 0.4\}$.

The theoretical positive equilibrium point will be $\overline{y} = 0.881024967$.

Figure.2. shows the behavior of the equilibrium point of the $y_{n+1} = \frac{3+2y_n}{1+y_n+4y_{n-3}}$. Which shows that the equilibrium point $\overline{y}$ is globally asymptotically stable, as we have shown theoretically.

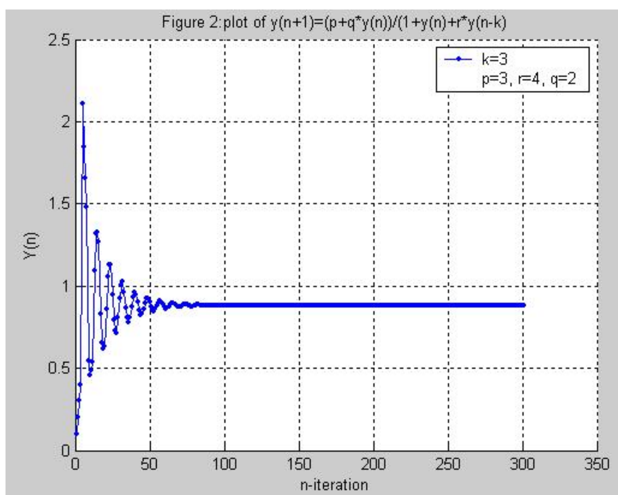


Table.2. shows that the numerically equilibrium point $\overline{y} = 0.8810$.

N	X(n)	N	X(n)	N	X(n)	N	X(n)
1	0.1000	26	0.7938	51	0.8770	276	0.8810
2	0.2000	27	0.7283	52	0.8540	277	0.8810
3	0.3000	28	0.7151	53	0.8475	278	0.8810
4	0.4000	29	0.8071	54	0.8586	279	0.8810
5	2.1111	30	0.9261	55	0.8790	280	0.8810
6	1.8466	31	1.0027	56	0.8986	281	0.8810
7	1.6540	32	1.0292	57	0.9071	282	0.8810
8	1.4828	33	0.9621	58	0.9013	283	0.8810
9	0.5459	34	0.8690	59	0.8865	284	0.8810
10	0.4581	35	0.8058	60	0.8709	285	0.8810
11	0.4850	36	0.7786	61	0.8623	286	0.8810
12	0.5353	37	0.8099	62	0.8641	287	0.8810
13	1.0945	38	0.8740	63	0.8739	288	0.8810
14	1.3214	39	0.9315	64	0.8862	289	0.8810
15	1.3241	40	0.9637	65	0.8945	290	0.8810
16	1.2649	41	0.9470	66	0.8950	291	0.8810
17	0.8324	42	0.8992	67	0.8886	292	0.8810
18	0.6554	43	0.8530	68	0.8792	293	0.8810
19	0.6201	44	0.8245	69	0.8719	294	0.8810
20	0.6348	45	0.8283	70	0.8701	295	0.8810
21	0.8600	46	0.8584	71	0.8739	296	0.8810
22	1.0532	47	0.8949	72	0.8807	297	0.8810
23	1.1264	48	0.9224	73	0.8869	298	0.8810
24	1.1259	49	0.9253	74	0.8894	299	0.8810
25	0.9435	50	0.9052	75	o.8874	300	0.8810

Table 2: Solution of DE $y_{n+1} = \frac{3+2y_n}{1+y_n+4y_{n-3}}$

Example 3.3. *Assume Eq.(3.2) holds. take $k = 3$, $p = 4$, $r = 1$ and $q = 5$. So the equation will be*

$$y_{n+1} = \frac{4 + 5y_n}{1 + y_n + y_{n-3}}, \; n = 0, 1, 2, \cdots$$

We assumed that the initial points $\{x_0, x_1, x_2, x_3\}$ all to be $\in (0, \infty)$ and are $\{0.1, 0.2, 0.3, 0.4\}$.

The theoretical positive equilibrium point will be $\overline{y} = 2.732050808$.

Figure.3. shows the behavior of the equilibrium point of the $y_{n+1} = \frac{4+5y_n}{1+y_n+y_{n-3}}$. Which shows that the equilibrium point $\overline{y}$ is globally asymptotically stable, as we have shown theoretically.

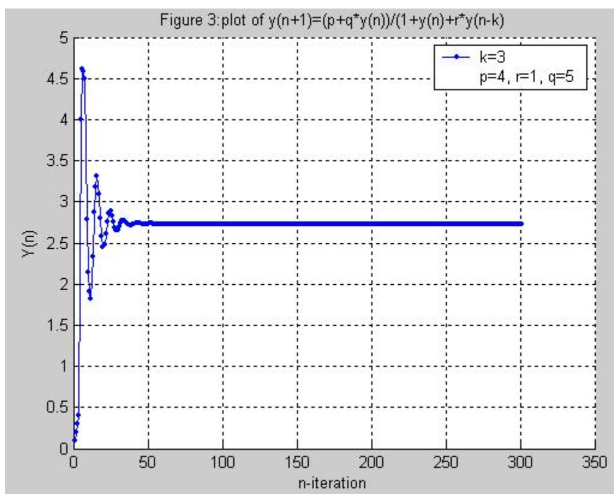


Table.3. shows that the numerically equilibrium point $\overline{y} = 2.7321$.

N	X(n)	N	X(n)	N	X(n)	N	X(n)
1	0.1000	26	2.8351	51	2.7349	276	2.7321
2	0.2000	27	2.7572	52	2.7356	277	2.7321
3	0.3000	28	2.6859	53	2.7346	278	2.7321
4	0.4000	29	2.6522	54	2.7328	279	2.7321
5	4.0000	30	2.6607	55	2.7311	280	2.7321
6	4.6154	31	2.6961	56	2.7302	281	2.7321
7	4.5774	32	2.7390	57	2.7303	282	2.7321
8	4.4981	33	2.7687	58	2.7311	283	2.7321
9	2.7890	34	2.7753	59	2.7321	284	2.7321
10	2.1352	35	2.7624	60	2.7328	285	2.7321
11	1.9029	36	2.7397	61	2.7331	286	2.7321
12	1.8260	37	2.7193	62	2.7328	287	2.7321
13	2.3384	38	2.7094	63	2.7323	288	2.7321
14	2.8668	39	2.7113	64	2.7318	289	2.7321
15	3.1777	40	2.7215	65	2.7315	290	2.7321
16	3.3127	41	2.7337	66	2.7316	291	2.7321
17	3.0918	42	2.7422	67	2.7318	292	2.7321
18	2.7964	43	2.7444	68	2.7321	293	2.7321
19	2.5784	44	2.7408	69	2.7323	294	2.7321
20	2.4513	45	2.7344	70	2.7323	295	2.7321
21	2.4845	46	2.7286	71	2.7323	296	2.7321
22	2.6147	47	2.7256	72	2.7321	297	2.7321
23	2.7569	48	2.7261	73	2.7320	298	2.7321
24	2.8647	49	2.7290	74	2.7319	299	2.7321
25	2.8859	50	2.7324	75	2.7319	300	2.7321

Table 3: Solution of DE $y_{n+1} = \frac{4+5y_n}{1+y_n+y_{n-3}}$

Chapter 4
Special Cases α β A B $C = 0$

4 Special Cases $\alpha\ \beta\ A\ B\ C = 0$

In this chapter we examine the character of solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}} \tag{4.1}$$

where one or more of the parameters in Eq.(4.1)are zero.

Observe that some of these equations are meaningless like the case when the parameters in the denominator are zero, and some of them are quite interesting and have been studied by many researchers.

If we assume the parameters α, β, A, B, C to be nonnegative, then Eq.(4.1) contains, as special cases, 21 difference equations with positive parameters. One of them is Eq.(4.1). Of the remaining 20 equations, some equations are trivial, Linear, or reducible to linear, or of the Riccati type

$$y_{n+1} = \frac{ay_n + b}{cy_n + d},\ n = 0, 1, \cdots$$

with nonnegative parameters a, b, c, d which itself is reducible to a linear equation by a well known change of variables.

Now we will mention the 20 equations with some details about each one.

4.1 One parameter $= 0$

In this section we examine the character of solution of Eq.(4.1) where one parameter in Eq.(4.1) $= 0$. There are five such equations, namely:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n}, n = 0, 1, 2... \tag{4.2}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Cx_{n-k}}, n = 0, 1, 2... \tag{4.3}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Cx_{n-k}}, n = 0, 1, 2... \tag{4.4}$$

$$x_{n+1} = \frac{\alpha}{A + Bx_n + Cx_{n-k}}, n = 0, 1, 2... \tag{4.5}$$

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n + Cx_{n-k}}, n = 0, 1, 2... \tag{4.6}$$

where the parameters α, β, A, B, C are posititve real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0$ are arbitrary nonnegative real numbers.

The Eq.(4.2) was investigated in [14], which is in fact a Riccati equation.

The change of variables

$$x_n = \frac{A}{C} y_n$$

reduces Eq.(4.3) to the difference equation

$$y_{n+1} = \frac{p + qy_n}{1 + y_{n-k}}, n = 0, 1, \cdots \tag{4.7}$$

where

$$p = \frac{\alpha C}{A^2} \quad and \quad q = \frac{\beta}{A}$$

Eq.(4.7) has a unique positive equilibrium point $\overline{y}$ given by

$$\overline{y} = \frac{(q-1) + \sqrt{(q-1)^2 + 4p}}{2}$$

The Eq.(4.7) was investigated in [6]. The authors studied the global stability, boundedness of positive solutions, and character of semicycles of Eq.(4.7).

The change of variables

$$x_n = \frac{\beta}{B} y_n$$

reduces Eq.(4.4) to the difference equation

$$y_{n+1} = \frac{p + y_n}{y_n + qy_{n-k}}, n = 0, 1, 2... \tag{4.8}$$

where

$$p = \frac{\alpha B}{\beta^2} \quad and \quad q = \frac{C}{B}$$

with $p, q \in (0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_0$ are nonnegative real numbers. Eq.(4.8) was investigated in [5]. They have concentrated on invariant intervals, the character of semicycles, the global stability, and the boundedness.

The change of variables

$$x_n = \frac{\alpha}{A} y_n$$

reduces the equation Eq.(4.5) to the difference equation

$$y_{n+1} = \frac{1}{1 + py_n + qy_{n-k}}, n = 0, 1, 2, \cdots \tag{4.9}$$

where

$$p = \frac{\alpha B}{A^2} \quad and \quad q = \frac{\alpha C}{A^2}$$

The unique positive equilibrium point of Eq.(4.9) is

$$\overline{y} = \frac{-1 + \sqrt{1 + 4(p+q)}}{2(p+q)}$$

we can show easily that this equilibrium point is locally asymptotically stable, for all values of parameters, and that Eq.(4.9) has no prime period two solutions.

By applying linearized stability and theorem(2.13), we can also show easily that this positive equilibrium point of Eq.(4.9) is globally asymptotically stable.

The change of variables

$$x_n = \frac{A}{C} y_n$$

reduces Eq.(4.6) to the difference equation

$$y_{n+1} = \frac{y_n}{1 + py_n + qy_{n-k}}, \; n = 0, 1, \cdots \tag{4.10}$$

where

$$p = \frac{\beta}{A} \quad and \quad q = \frac{B}{C}$$

Eq.(4.10) always has zero as an equilibrium point, and when $p > 1$, it also has the unique positive equilibrium point

$$\overline{y} = \frac{p-1}{q+1}$$

The following results are a straight forward consequence of the theorem(2.21) of the global asymptotic stability of the zero equilibrium point and theorem(2.22).

Theorem 4.1. *Assume $p \leq 1$, then the zero equilibrium of Eq.(4.10) is globally asymptotically stable.*

Theorem 4.2. *Assume that $y_{n-k}, y_{n-k+1}, \cdots, y_{-1}, y_0 \in (0, \infty)$ and $p > 1$ then the positive equilibrium of Eq.(4.10) is globally asymptotically stable.*

4.2 Two parameters are zero

In this section we examine the character of solutions of Eq.(4.1) where two parameters in Eq.(4.1) are zero. There are nine such equations, namely:

$$x_{n+1} = \frac{\alpha}{A + Bx_n}, n = 0, 1, 2... \tag{4.11}$$

$$x_{n+1} = \frac{\alpha}{A + Cx_{n-k}}, n = 0, 1, 2... \tag{4.12}$$

$$x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-k}}, n = 0, 1, 2... \tag{4.13}$$

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n}, n = 0, 1, 2... \tag{4.14}$$

$$x_{n+1} = \frac{\beta x_n}{A + Cx_{n-k}}, n = 0, 1, 2... \tag{4.15}$$

$$x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-k}}, n = 0, 1, 2... \tag{4.16}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{A}, n = 0, 1, 2... \tag{4.17}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n}, n = 0, 1, 2... \tag{4.18}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{Cx_{n-k}}, n = 0, 1, 2... \tag{4.19}$$

where the parameters α, β, A, B, C are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0$ are arbitrary nonnegative real numbers.

Two of these equations, namely Eq.(4.11) and Eq.(4.14) are Riccati- type difference equation.

Its interesting to note that the change of variables

$$x_n = \frac{1}{y_n}$$

reduces the Riccati equation Eq.(4.14) to the linear equation

$$y_{n+1} = \frac{A}{\beta} y_n + \frac{B}{\beta}, \ n = 0, 1, \cdots$$

for which the global behavior of solutions is easily derived.

Eq.(4.12) is essentially Riccati equation. Indeed if $\{x_n\}$ is a solution of Eq.(4.12), then the subsequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ satisfy the Riccati equation of the form of Eq.(4.11).

Now consider the equation

$$x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-k}}$$

The Eq.(4.13) which by change of variables

$$x_n = \frac{\sqrt{\alpha}}{y_n}$$

reduces to the difference equation

$$y_{n+1} = \frac{B}{y_n} + \frac{C}{y_{n-k}}, n = 0, 1, 2, ... \tag{4.20}$$

where the initial conditions $y_{-k}, \cdots, y_0$ are arbitrary nonnegative real numbers.

The only positive equilibrium point is $\overline{y} = \sqrt{B + C}$. When $k = 1$, Eq.(4.20) was investigated in [14]. It was shown that every solution is bounded, it also shown that the equilibrium point

$$\overline{y} = \sqrt{B + C}$$

is globally asymptotically stable.

In this monograph, we investigate the difference Eq.(4.20) when $k \in \{2, 3, ...\}$.

Theorem 4.3. *Every solution of Eq.(4.20) is bounded.*

Proof. Assume for the sake of contradiction that there exists a solution $\{y_n\}_{n=-k}^{\infty}$ which is neither bounded from above nor from below. That is

$$\lim_{n\to\infty} sup\ y_n = \infty \ and\ \lim_{n\to\infty} inf\ y_n = 0$$

then clearly, we can find indices i and j with

$$1 \leq i < j$$

such that

$$y_i > y_n > y_j \; for \; all \; n \in \{-k, ..., j-1\}$$

Hence

$$y_j = \frac{B}{y_{j-1}} + \frac{C}{y_{j-k-1}} > \frac{B+C}{y_i}$$

and

$$y_i = \frac{B}{y_{i-1}} + \frac{C}{y_{i-k-1}} \leq \frac{B+C}{y_j}$$

that is

$$B + C < y_i y_j < B + C$$

which is impossible.

□

To investigate the stability of Eq.(4.20), let $f(x, y) = \frac{B}{x} + \frac{C}{y}$.

Theorem 4.4. *The equilibrium point $\overline{y} = \sqrt{B+C}$ is unstable when k is even.*

Proof. The linearized equation of Eq.(4.20) about the equilibrium point

$$\overline{y} = \sqrt{B+C}$$

is

$$z_{n+1} = -\frac{B}{B+C} z_n - \frac{C}{B+C} z_{n-k}, \; n = 0, 1, 2, ...$$

and its characteristic equation is

$$\lambda^{k+1} + \frac{B}{B+C}\lambda^k + \frac{C}{B+C} = 0$$

Then the proof follows immediately from the linearized stability theorem(2.12).

□

Before we examine the existence of two cycles of Eq.(4.20), it is worthwhile to mention that when $C = 1$ *and* $k = 2$, it was shown by *R.Devault* and *G.Ladas* and *S.W. Schultz* in [8], that every positive solution of the difference equation

$$y_{n+1} = \frac{B}{y_n} + \frac{1}{y_{n-2}}$$

converges to a period two solution.

Theorem 4.5. *The Eq.(4.20) has prime period two solution if and only if k is even.*

Proof. Let

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

be a period two solution of the Eq.(4.20), then

- If k is odd, then

  $$\Phi = \frac{B}{\Psi} + \frac{C}{\Phi} \; and \; \psi = \frac{B}{\Phi} + \frac{C}{\Psi}$$

  thus

  $$\phi = \psi$$

  which is contradiction.

- If k is even, then

  $$\Phi = \frac{B}{\Psi} + \frac{C}{\Psi} \; and \; \psi = \frac{B}{\Phi} + \frac{C}{\Phi}$$

  which implies that

  $$\phi\psi = B + C$$

  and the period two solution must be of the form

  $$..., \phi, \frac{B + C}{\phi}, \phi, \frac{B + C}{\phi}, ...$$

  which completes the proof.

□

Theorem 4.6. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(4.20). Then the following statements are true:*

1. *Suppose that $B+C > 1$ and assume that for some $N \geq 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [1, B+C]$. Then $y_n \in [1, B+C]$ for all $n > N$.*

2. *Suppose that $B+C < 1$ and assume that for some $N \geq 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [B+C, 1]$. Then $y_n \in [B+C, 1]$ for all $n > N$.*

3. *Suppose that $B > C$ and assume that for some $N \geq 0$,$y_{N-k}, ..., y_{N-1}, y_N \in [C, \frac{B}{C}+1]$. Then $y_n \in [C, \frac{B}{C}+1]$ for all $n > N$.*

4. *Suppose that $B < C$ and assume that for some $N \geq 0$,$y_{N-k}, ..., y_{N-1}, y_N \in [B, \frac{C}{B}+1]$. Then $y_n \in [B, \frac{C}{B}+1]$ for all $n > N$.*

Proof. The proof of this theorem is based on monotonic character.

1. Assume that for some $N > 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [1, B+C]$. Then

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \leq B+C$$

and

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \geq \frac{B}{B+C} + \frac{C}{B+C} = 1$$

2. Assume that for some $N > 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [B+C, 1]$. Then

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \leq \frac{B}{B+C} + \frac{C}{B+C} = 1$$

and

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \geq B+C$$

3. Assume that for some $N > 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [C, \frac{B}{C}+1]$. Then

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \leq \frac{B}{C} + \frac{C}{C} = \frac{B}{C} + 1$$

and

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \geq \frac{B}{\frac{B+C}{C}} + \frac{C}{\frac{B+C}{C}} = C$$

4. Assume that for some $N \geq 0$, $y_{N-k}, ..., y_{N-1}, y_N \in [B, \frac{C}{B} + 1]$. Then

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \leq \frac{B}{B} + \frac{C}{B} = \frac{C}{B} + 1$$

and

$$y_{N+1} = \frac{B}{y_N} + \frac{C}{y_{N-k}} \geq \frac{B}{\frac{B+C}{B}} + \frac{C}{\frac{B+C}{B}} = B$$

The proof is complete.

□

Theorem 4.7. *Let k be odd, then $\overline{y} = \sqrt{B+C}$ is globally asymptotically stable equilibrium point of Eq.(4.20).*

Proof. For $u, v \in (0, \infty)$, set

$$f(u, v) = \frac{B}{u} + \frac{C}{v}$$

Then $f : (0, \infty) \times (0, \infty) \to (0, \infty)$ is continuous function and is nonincreasing in both its argument. Let $(m, M) \in (0, \infty)$ is a solution of the system

$$m = f(M, M) \textit{ and } M = f(m, m)$$

then $m = M$ **when k is odd**. By using Theorem(2.18), $\overline{y} = \sqrt{B+C}$ is globally asymptotically stable equilibrium point of Eq.(4.20). This completes the proof. □

Finally, we introduce the analysis od semicycles of Eq.(4.20) in the following theorem.

Theorem 4.8. *Every oscillatory solution of Eq.(4.20) has semicycle of length at most k*

Proof. The proof follows from theorem (2.19) by observing that the function $f(u, v) = \frac{B}{u} + \frac{C}{v}$ is decreasing in both its arguments.
Thus the proof is complete.

□

The change of variables

$$x_n = \frac{A}{C}$$

reduces Eq.(4.15) to **Pielou's Difference Equation**

$$y_{n+1} = \frac{py_n}{1 + y_{n-k}} \tag{4.21}$$

where

$$p = \frac{\beta}{A}$$

when

$$p \leq 1$$

It follows from Eq.(4.21) that every solution converges to 0.
Furthermore, 0 is locally asymptotically stable when $p \leq 1$ and unstable when $p > 1$.

When

$$p > 1$$

the zero equilibrium of Eq.(4.21) is unstable and possesses the unique positive equilibrium

$$\overline{y} = p - 1$$

which is locally asymptotically stable.

We can obtain from the above the following theorem:

Theorem 4.9. *1. Assume*

$$p \leq 1$$

Then the zero equilibrium of Eq.(4.21) is globally asymptotically stable.

2. Assume $y_0 \in (0, \infty)$ and

$$p > 1$$

Then the positive equilibrium

$$\overline{y} = p - 1$$

of Eq.(4.21) is globally asymptotically stable.

The Eq.(4.16) which is by the change of variables

$$x_n = \frac{\beta}{Cy_n}$$

reduces it to the difference equation

$$y_{n+1} = P + \frac{y_n}{y_{n-k}}, n = 0, 1, 2, ... \tag{4.22}$$

where

$$P = \frac{B}{C}$$

and the initial conditions $y_{-k}, \cdots, y_0$ are nonnegative real numbers. The Eq.(4.22) was studied in [2].

Eq.(4.17) is linear .

Eq.(4.18) is a Riccati equation, its solved explicitly to determine the character of its solution in [10], they showed that the equilibrium point is globally asymptotically stable.

Eq.(4.19) which by the change of variables

$$x_n = \frac{\beta}{C}y_n$$

reduces to the difference equation

$$y_{n+1} = \frac{P + y_n}{y_{n-k}} \tag{4.23}$$

where

$$P = \frac{\alpha C}{\beta^2} \in (0, \infty)$$

and the initial conditions $y_{-k}, \cdots, y_0$ are arbitrary nonnegative real numbers.

The unique positive equilibrium point is

$$\overline{y} = \frac{1 + \sqrt{1 + 4p}}{2}$$

The linearized equation about equilibrium point $\overline{y}$ is

$$z_{n+1} - \frac{2}{1+\sqrt{1+4p}} z_n + z_{n-k} = 0$$

and its characteristic equation is:

$$\lambda^{k+1} - \frac{2}{1+\sqrt{1+4p}} \lambda^k + 1 = 0$$

.

Remark: For $k = 1$, the Eq.(4.19) is well known in literature of rational difference equations as **lyness' Equation** [14].

Theorem 4.10. *The equilibrium point* $\overline{y} = \frac{1+\sqrt{1+4p}}{2}$ *is unstable.*

Proof. The proof follow immediately by theorem(2.14).

□

Theorem 4.11. *The Eq.(4.23) has no positive prime period two solution.*

Proof. Assume for the sake of contradiction that there exists a solution of prime period two

$$..., \phi, \psi, \phi, \psi, ...$$

where ϕ and ψ are positive and distinct.

- If k is odd. Then we have

$$\phi = \frac{p+\psi}{\phi} \tag{4.24}$$

and

$$\psi = \frac{p+\phi}{\psi} \tag{4.25}$$

from Eq.(4.24), we get

$$\phi^2 = p + \psi \tag{4.26}$$

and from Eq.(4.25), we get

$$\psi^2 = p + \phi \tag{4.27}$$

from Eq.(4.26)and Eq.(4.27), we get

$$\phi + \psi = -1$$

which is a contradiction.

- If k is even. Then we have

$$\phi = \frac{p+\psi}{\psi} \tag{4.28}$$

and

$$\psi = \frac{p+\phi}{\phi} \tag{4.29}$$

from Eq.(4.28), we get

$$\phi\psi = p+\psi \tag{4.30}$$

and from Eq.(4.29), we get

$$\phi\psi = p+\phi \tag{4.31}$$

from Eq.(4.30)and Eq.(4.31), we get

$$p+\phi = p+\psi$$

hence

$$\psi = \phi$$

which is a contradiction.

This completes the proof. □

Theorem 4.12. *The equilibrium point* $\overline{y} = \frac{1+\sqrt{1+4p}}{2}$ *of Eq.(4.23) is globally asymptotically stable.*

Proof. For $u, v \in (0,\infty)$, set $f(u,v) = \frac{p+u}{v}$. Then

$$f : (0,\infty)\times(0,\infty) \to (0,\infty)$$

is continuous function and is nondecreasing in u and nonincreasing in v. Let $(m, M) \in (0,\infty)$ is a solution of the system

$$m = f(m,M) \ and \ M = f(M,m)$$

Then

$$p+m = p+M$$

Hence

$$m = M$$

Then by using theorem(3.24),

$$\overline{y} = \frac{1+\sqrt{1+4p}}{2}$$

is globally asymptotically stable equilibrium point of Eq.(4.23).
This completes the proof. □

Theorem 4.13. *Every oscillatory solution of Eq.(4.23) has semisycle of length at least* $k+1$.

Proof. The proof follows immediately from theorem(2.20), by observing that the function $f(x,y) = \frac{p+x}{y}$ is increasing in x and decreasing in y. The proof is complete. □

4.3 Three Parameters are Zero

In this section we examine the character of solution of Eq. (4.1) where three parameters in Eq.(4.1) are zero. There are 6 such equations, namely:

$$x_{n+1} = \frac{\alpha}{A}, \; n = 0, 1, 2... \tag{4.32}$$

$$x_{n+1} = \frac{\alpha}{Bx_n}, \; n = 0, 1, 2... \tag{4.33}$$

$$x_{n+1} = \frac{\alpha}{Cx_{n-k}}, \; n = 0, 1, 2... \tag{4.34}$$

$$x_{n+1} = \frac{\beta x_n}{A}, \; n = 0, 1, 2... \tag{4.35}$$

$$x_{n+1} = \frac{\beta x_n}{Bx_n}, \; n = 0, 1, 2... \tag{4.36}$$

$$x_{n+1} = \frac{\beta x_n}{Cx_{n-k}}, \; n = 0, 1, 2... \tag{4.37}$$

Eq.(4.32) and Eq.(4.36) both are trivial, since both of them are constants.

Eq.(4.33) was studied in [14], *G.LADAS* showed that every solution of Eq.(4.33) is periodic with period two.

Eq.(4.34) has nontrivial solution, and every solution is periodic with period $2(k+1)$. Eq.(4.35) is a linear difference equation.

Finally, Eq.(4.37), which using the change of variables,

$$x_n = \frac{\beta}{C} y_n$$

can be reduced to the difference equation

$$y_{n+1} = \frac{y_n}{y_{n-k}} \tag{4.38}$$

When $k = 1$, every solution of Eq.(4.38) is periodic with period 6, and its solution is:

$$\cdots, x_{-1}, x_0, \frac{x_0}{x_{-1}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{x_{-1}}{x_0}, \cdots$$

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Chapter 5
The Matlab 6.5 Code

5 The Matlab Code 6.5

Birzeit University
Muna Abu Al-Halaweh , (1075342)
Professor: Mohammad Saleh

Matlab code for the fixed point graph

```
clear all
dt=0.01;
a=1;
b=4;
n=(b-a)/dt;
t=a:dt:b;
for i=1:n+1
```
$y(i) = t(i)^2 - 4 * t(i) + 6;$
```
end
```
plot(t,y,t,t) grid on title('the equilibrium points for $f(x) = x^2 - 4x + 6'$).

Matlab code for the cobweb diagram for some μ {Figures A and B}

```
clear all
m= input('insert the value of m=');
x(1)=input('give initial value x(0) to find he following iteration=')
dt=0.01;
a=0;
b=1;
n=(b-a)/dt;
t=a:dt:b;
for i=1:n+1
y(i)= m*t(i)*(1-t(i));
end grid on plot(t,y,t,t)

number=20;
hold on
for i=1:number
x(i+1)= m*x(i)*(1-x(i));
line([x(i) x(i+1)],[ x(i+1) x(i+1)])
line([x(i) x(i)],[ x(i) x(i+1)])
```

```
end
' n x(n)'
i=1;
[i x(i)]
for i=2:number
if(x(i)==x(i-1))
break
end
[i x(i)]
end
```

Matlab code for a general solution of the rational difference equation of order k, this program solves the equation:
Yn+1 = (p + q*Yn)/(1+ Yn+r*Yn-k)

```
function ratdiff;
k=input('enter the value of the positive integer k = ');
p=input('enter the value of the positive parameter p = ');
q=input('enter the value of the positive parameter q = ');
r=input('enter the value of the positive parameter r = ');

    solution =ddifkk(k,p,q,r);
disp(' ')
disp(' Table ')
disp(' ')
disp('The solution x(n) is given in the following table : ')
d=[solution(1:25,:),solution(26:50,:),solution(51:75,:),solution(276:300,:)];
disp('————————————————————————————————————-')
disp(' n x(n) n x(n) n x(n) n x(n) disp('——————————————————————
——————————-')
disp(d)
```

fixedpoint=$(((q-1)+sqrt((q-l)^2+4*p*(r+1)))/(2*(r+1)))$;

```
fprintf('fixedpoint =function plotandeval=ddifkk(k,p,q,r);

for i=1:k+1;
x(i)=input('enter the value of the positive initial condition x=');
end

for n=k+1:300;
```

```
x(n+1)=(p+q*x(n))/(1+x(n)+r*x(n-k));
end

t=1:301;
plotandeval=[t;x]';
grid on
hold on

t=1:301;
plot(t,x,'b.-');
xlabel('n-iteration');
ylabel('Y(n)');
title('Figure :plot of y(n+1)=(p+q*y(n))/(1+y(n)+r*y(n-k)');
p1=strcat('k= ',num2str(k));
p2=strcat('p= ',num2str(p),', r= ,num2str(r),', q= ',num2str(q));
legend(p1,p2);
```

References

[1] Abu-Baha'.S, Dynamics of a k^{th}order Rational Difference Equation Using Theoretical and Computational Approach.

[2] Abu-Saris.R, DeVault.R, Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. 16(2003) 173-178.

[3] Amleh.A.M, Camouzis.E and Ladas.G, On second order rational difference equations, Part 1, Open Problems and Conjectures, Tournal of difference equations and applications, 13(11) (2007) 969-1004.

[4] Dehghan.M, Douraki.M.J, The Oscillatory Character of the Recursive Sequence $x_{n+1} = \frac{\alpha+\beta x_{n-k+1}}{A+Bx_{n-2k+1}}$, Applied mathematics and Computation, 175 (2006) 38-48.

[5] Dehghan.M, etal. Dynamics of a rational difference equations using both theoretical and computational approaches, Applied Mathematics and Computation, 168 (2005) 756-775.

[6] Dehghan.M, Sebdani.R.M, Dynamics of a higher rational difference equation, Appl. Math. Comp. 178(2006)345-354.

[7] Devault.R, Kosmala.W, Ladas.G, Schuults.S.W, Global behavior of $y_{n+1} = \frac{p+y_{n-k}}{qy_n+y_{n-k}}$, Nonlinear Analysis, 47 (2001) 4743-4751.

[8] Devalut.R, Shultz.S.W and Ladas.G, On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, Proceedings of the American Mathematical Society, 126(1998) 3257-3261.

[9] Douraki.M.J, Dehghan.M, Razzaghi.M, The qualitative behavior of solutions of a non-linear difference equation, Appl. Math., Comp. 170(2005) 485-502.

[10] El-Afifi.M.M, On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}$,Applied Mathematics and Computations, 147 (2004) 617-628.

[11] Elaydi.S, Introduction to difference equations, Springer-NewYork, 1996.

[12] Knoph.P.M and Huang.Y.S, On the boundedness characters of some rational difference equations, Journal of difference equations and Application, 14(7) (2008) 769-777.

[13] Kocic.V.L and Ladas.G, Global Asymptotic Behavior on Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrect, 1993.

[14] Kulenovic.M.R.S, Ladas.G, Dynamics of the Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Raton,2001.

[15] Kulenovic.M.R.S, Ladas.G, Martins.L.F, and Rudrigues.I.W, The Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}$, Facts and Conjectures,computers and mathematics with applications, 45 (2003) 1087-1099.

[16] Kuruklis.S.A,The Asymptotic Stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, Journal of difference equation and application, 188 (1994) 719-731.

[17] Li.W.T, Sun.H.R, Dynamics of a Rational difference equation, Applied Mathematics and Computations, 163 (2005) 577-591.

[18] Saleh.M, Aloqeili.M, On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, Appl. Math. Comp. 171(2005) 862-869.

[19] Saleh.M, Aloqeili.M, On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Comp. 177(2006) 189-193.

[20] Saleh.M, Abu-baha', Dynamics of a higher order difference equation, Applied Mathematics and computation, 181 (2006) 84-102.

[21] Su.Y.H, Li.W.T, Global asymptotic stability of a second order nonlinear difference equation, Applied Mathematics and Computation, 168 (2005) 981-989.

[22] Yau.X.X, Li.W.T, Zhao.Z, Global asymptotic stability for a higher order nonlinear difference equations, Applied Mathematics and computation, 182 (2006) 1819-1831.